Modeling of a horizontal asymmetric U-shaped vibration-based piezoelectric energy harvester (U-VPEH)

Shilong Sun, Peter W. Tse

Department of Systems Engineering and Engineering Management (SEEM), City University of Hong Kong, Hong Kong, China

**Abstract**

In this paper, we present a new horizontal asymmetric U-shaped vibration-based piezoelectric energy harvester (U-VPEH), which collects and converts destructive vibration energy into useful electrical energy. The finite element analysis is conducted on modal shape, the displacement, the mechanical energy, and the strain for both the linear and nonlinear U-VPEH models. Mathematical governing equations are derived to investigate dynamic characteristics of the model. The harmonic balance method and state space form are utilized to conduct the analytical and theoretical analysis. The results show that the first and second eigenfrequencies are 17.167 Hz and 22.951 Hz for the nonlinear U-VPEH model while those are only 17.366 Hz and 25.124 Hz for the linear model. The nonlinear energy harvester model can narrow frequency band gap between the first two resonance modes. Moreover, the maximum voltage response is 8.743 V@16.5 Hz under the up-sweeping signals while the maximum voltage response is 14.18 V@15.41 Hz under the down-sweeping signals. The experimental results demonstrate that the voltage response and the resonance frequency of the U-VPEH agree with the analytical and theoretical analysis. The nonlinear horizontal asymmetric U-VPEH model exhibits a good performance on the energy transfer. This higher energy output, lower resonance frequency, and closer resonance peak can broaden the flexibility and practical usage of the energy harvester.

**1. Introduction**

In recent decades, the research in vibration-based energy harvester via piezoelectric transducers has been attracting intensive attention in the research sector of sustainable energy [1]. The piezoelectric is widely used to convert the mechanical energy into useful electric energy on electromechanical systems due to its advantages of high power density and easy implementation on mechanical devices [2,3]. The mechanical energy has a broader usage and application in automobiles, buildings, railways, and windmills. The single-degree-of-freedom (SDOF) vibration energy harvester model has been widely studied by numerous researchers because of its simple geometric structure and high efficiency of energy transformation [4–7]. However, the shortcomings of the traditional vibration energy harvester, including the sole resonance mode, narrow bandwidth and insufficient power output, are very common [8]. As the alternative approaches to improve the performance of energy harvesters, the model nonlinearity and multimodality are proposed and studied, such as geometric nonlinear structure systems [9], nonlinear magnetic effect systems [10], and X-stable [11–13] systems.
The nonlinearity technology is an effective method to achieve the optimal design of the broadband energy harvester and also shape the resonance frequency spectrum of the energy harvester to better suit the ambient vibration [14,15]. Generally, the technology compromises two parts, i.e., the geometric nonlinear structure and the magnetic effect nonlinearity design implementation. The geometric nonlinear structure technology for a vibration-based energy harvester design has been analyzed and reported recently. The vibration-based piezoelectric energy harvester based on the different geometric structures are classified into the M-shaped [16,17], L-shaped [18,19], H-shaped [20], and also variable-shaped [21]. Erturk [16,17] demonstrated an M-shaped asymmetric nonlinear structure which could be exploited and utilized into the piezoelectric and electromagnetic energy harvester due to its large strain and kinetic energy regions. Asan [22] optimized the piezoelectric beam shape to improve the bandwidth and the power output of the broadband vibration energy harvester. Yang developed an high-efficiency compressive-mode piezoelectric energy harvester (HC-PEH) [23] and an arc-shaped piezoelectric elements energy harvester [24]. The HC-PEH exhibited the favorable nonlinear phenomena at low frequency range and generated a maximum power 19 mW@ 21 Hz. Ansari [25] investigated a fan-folded vibration energy harvester for leadless pacemakers and can generate more than 10 μW of power per cubic centimeter. Abdelmoula [26] designed a Zigzag energy harvester for low-frequency devices, of which the results opened new opportunities for lower frequency energy harvester design without increasing its geometric dimensions. Zhou [27] developed a flexible longitudinal zigzag structure for enhanced vibration energy harvesting and can improve the energy conversion efficiency at low frequencies. All the aforementioned research works were conducted on the basis of the geometric nonlinear structure field, the system nonlinearity was achieved through the changes of the proposed structure shape. However, it is not sufficient to achieve the broadband energy harvester design only by the structure shape, because the shape of the structure cannot be adjusted once they are implemented on the rotational machines. This is one of the main drawbacks that lead to inconvenience in harnessing the energy from the ambient environment.

Another common technique of the realization of nonlinearity into the vibration-based energy harvester design is exploiting the magnets interaction. The nonlinear magnets interaction coupling is popular in the employment of achieving the broadband energy harvesting. Cao [28] investigated the nonlinear dynamic characteristics of a magnetically coupled piezoelectric energy harvester, where the angle of the magnets was adjustable and the rotating magnets could produce nonlinear adjustable magnetic force. Besides, Deng [29,30] also reported a nonlinear electromagnetic energy harvester with various magnetic orientation and Zhou [31] investigated a broadband piezoelectric energy harvesting using rotatable magnets with altering the angular orientation of the external magnets. The results indicate that adjust the magnets orientation and the magnets angular can achieve the broadband frequency. Fan [32] addressed a compact bi-directional nonlinear piezoelectric energy harvester (PEH) which composed of two magnets coupling cantilevered beams at two orthogonal directions. The results showed that the introduction of the magnetic nonlinearity could not only enable the energy transfer between two beams, but also improve the efficiency of the PEH at low-level excitation. Salauddin [33] proposed a new design of hybridized electromagnetic energy harvester which collected the human-body motion energy efficiently. Wu [34] reported a broadband nonlinear two-degree-of-freedom piezoelectric energy harvester and achieved a significantly wider bandwidth. Wang [35] proposed a bi-stable two-degree-of-freedom energy harvester with magnetic coupling to improve the performance of operating bandwidth. However, the nonlinearity of the magnet effect is inflexible in practice, especially when the excitation is under the low-level amplitude. Therefore, the combination of the nonlinearity of geometric and the magnetic effect needs more efforts to be explored with the aim of shifting the resonance frequency and expanding the power output bandwidth by designing the piezoelectric energy harvester.

In addition to the nonlinearity methods, the multimodality can broaden the bandwidth and enlarge the power output frequency range. Most multimodal energy harvesters are designed on the basis of the cantilever beam theory which can produce close vibration resonance mode to make the frequency bandwidth enlarged. Han [36] and Tang [37] both proposed a multi-degree-of-freedom piezoelectric vibration energy harvester which showed that such systems could extract more energy than the single-degree-of-freedom energy harvester. Meanwhile, they can also offer close resonance frequency and keep the power output at large amplitude with nearby resonance frequencies. Wu [38] designed a 3-DOF resonance energy harvester studying the dynamic characteristics of human motion and collecting an average generated power of 2.28 mW. Mohamed [39] conducted the research on the multimodal vibration energy harvester on the plate structures and the natural frequencies were in the range from 8 to 19 Hz. Sun [40] also presented a multimodal vibration-based energy harvester model for the low-frequency rotational machines which covered a frequency range from 18.18 Hz to 26.8 Hz. The aforementioned studies mainly focus on the linear multimodality implement application when designing the piezoelectric energy harvester.

In this paper, we present a new horizontal asymmetric U-shaped vibration-based piezoelectric energy harvester (U-VPEH), in which the nonlinearity is incorporated with multimodality. The mechanical part is designed based on the traditional cantilevered vibration energy harvester which exhibits multimodality characteristics when it is vibrating. This multimodality can produce a close multiple resonance frequencies band which cover the low range rotational machines’ frequencies and maximize the power output at the corresponding resonance frequency. The electric part consists of the piezoelectric and magnets. The piezoelectric material is used to collect and store the energy transferred from the mechanical fields. The magnets will exert the magnetic effect and can enhance the nonlinearity of the whole structure’s nonlinear performance further. Therefore, compared with previous research works on vibration-based piezoelectric energy harvester, this proposed U-VPEH exploits the advantages of geometric and magnetic nonlinearity and multimodality.

The reminder of the paper is organized as follows. Section 2 describes the design of the proposed U-shaped vibration-based piezoelectric energy harvester (U-VPEH) model. Simulation results through the finite element analysis, including
the displacement, the mechanical energy and the strain are provided in Section 3. Section 4 demonstrates the theoretical analysis and derives the governing equations for the proposed U-VPEH model. The results and discusses are given in Section 5. Section 6 summarizes the paper with concluding remarks.

2. Description of the U-VPEH model design

Fig. 1 shows the proposed U-VPEH model design. The left end of the beam is fixed and the right end is free. There are two substrate beams, called the primary beam and the auxiliary beam. The primary beam is longer than the auxiliary beam which results in the asymmetric behavior. The base excitation is assumed and given on the fixed end of the primary beam. Two tip masses are attached on the free end of the both beams. The permanent magnets are placed on the right area of the main structure which can produce the repelling force between the first tip mass and the magnets.

The U-VPEH model exhibits linear characteristics if we inactive the two permanent magnets on the right side. We give the base excitation on the fixed end along the direction of perpendicular to the surface of the beam. We defined the simulated excitation signal from 12 Hz to 26 Hz at a rate of 0.1 Hz/s. The external resistor is denoted as 12 kΩ. There are two piezoelectric materials laminated on the primary beam and the auxiliary beam, respectively. The first piezoelectric is fixed on the upper surface at the left end of the primary beam, while the second piezoelectric is glued on the lower surface in the right end of the auxiliary beam. The primary beam and the auxiliary beam consist of the aluminum while the tip mass is made of the iron.

3. Finite element analysis of the proposed U-VPEH model

3.1. The linear U-VPEH model

The finite element analysis is used to study the resonance frequency, the mode shape, the stress, the electric potential of the piezoelectric layer, and the mechanical energy. We conduct a simulation of the proposed linear model through the COMSOL Multiphysics software. The AC/DC, the electrical circuit, the solid mechanics and the magnetic fields are utilized in this simulation part. The results of the proposed model are expressed as follows.

As introduced before, the proposed U-VPEH model expresses a linear model after eliminating the magnets. Fig. 2 shows the mode shape analysis of the linear energy harvester. The first and second resonance frequencies are 17.366 Hz and 25.124 Hz, respectively. The bandwidth of the first two resonance frequencies is 7.758 Hz.

Fig. 3 demonstrates the finite element results of the piezoelectric layer on the linear U-VPEH model, including the displacement, mechanical energy, and mechanical strain. The displacement on the 1st and 2nd piezoelectric layer is shown.
Fig. 2. Mode shape analysis of the linear U-VPEH model: (a) the 1st resonance mode; (b) the 2nd resonance mode.

Fig. 3. Finite element results of piezoelectric layer on the linear U-VPEH model: (a) displacement; (b) mechanical energy; (c) strain.
in Fig. 3(a). The displacement is collected on the right end of the first piezoelectric and the left end of the second piezoelectric. The displacement of the first piezoelectric is smaller than that of the second piezoelectric under the whole swept frequencies. For the same measured point, the piezoelectric on the first resonance frequency is also smaller than the second resonance frequency. That means that the second vibration mode can produce larger deformation than the first vibration mode.

Fig. 3(b) shows the mechanical energy of the 1st and 2nd piezoelectric layer. The mechanical energy of the first piezoelectric is larger than that of the second piezoelectric at the first resonance frequency while the mechanical energy of the first piezoelectric is lower than that of the second piezoelectric at the second resonance frequency. The total mechanical energy at the 2nd resonance frequency is larger than the 1st resonance frequency for both two beams which is in agreement with the displacement performance shown in Fig. 3(a). Fig. 3(c) shows the strain along the piezoelectric layer. For the first resonance frequency, the strain on the primary beam is larger than that of the auxiliary beam. However, the strain on the primary beam is lower than that of the auxiliary beam at the second resonance vibration mode.

### 3.2. The nonlinear U-VPEH model

Fig. 4 shows the mode shape analysis of the nonlinear U-VPEH model. The first and second eigenfrequency 17.167 Hz and 22.951 Hz, respectively. The bandwidth between these two eigenfrequencies of the nonlinear U-VPEH model is 5.784 Hz, comparing to a bandwidth 7.758 Hz of the linear one. We can infer that the nonlinear vibration-based energy harvester can provide closer vibration peak which is convenient for the power output transfer. The effect of the magnetic field is investigated in the following section.

Fig. 5 shows the contour plot of the magnetic flux density for the two permanent magnets. Here, the strength of the permanent magnet definition utilizes the concept of magnetization and the magnetization is 750 kA/m. It can be found that the

![Mode shape analysis of the nonlinear U-VPEH model: (a) the 1st resonance mode; (b) the 2nd resonance mode.](image)

![Contour plot of the magnetic flux density distribution for the two permanent magnets.](image)
magnetic field lines are mostly distributed symmetrically along the central neutral layer of the tip mass. In other words, the magnetic force acting on the tip mass are also symmetrical. When it vibrates along the positive y-direction, the magnetic force between the upper magnet and the tip mass will increase while the magnetic force between the lower magnet and the tip mass will decrease. There is a similar trend when it vibrates along the negative y-direction because of the changes of the distance between these two blocks. The magnetic force can be decomposed into two sub magnetic force, one is along the y-direction and the other is along the x-direction.

Fig. 6 shows the displacement performance and the magnetic flux density of the nonlinear U-VPEH model at the first eigenfrequency. The displacement of the tip mass on the primary beam is much larger than that of the second one when the whole structure vibrating under the influence of the magnetic field at the first eigenfrequency. The red color represents the large displacement while the blue represents the small displacement. The maximum RMS displacement is around 0.76 mm and the maximum magnetic flux density is around 0.78 T. However, there is a little disadvantage that the maximum displacement of the nonlinear model is slightly smaller than the linear one. This is because the magnetic force acting on the tip mass makes the beam deformation smaller and thereby the tip displacement drops.

Fig. 7 exhibits the finite element results of the piezoelectric layer on the nonlinear U-VPEH model, including the displacement, the mechanical energy, and the strain. From Fig. 7(a), the displacement of the second piezoelectric is larger than the first piezoelectric under the whole swept frequencies. The first and second peak displacements are 239.06 mm@16.6 Hz and 281.74 mm@22.3 Hz, respectively, for the first piezoelectric. By comparison, for the second piezoelectric, the first and second peak displacements are 35.535 mm@16.6 Hz and 32.885 mm@22.3 Hz, respectively.

Fig. 7(b) shows the mechanical energy on the nonlinear U-VPEH model. The performance of the mechanical energy exhibits a consistent trend with the displacement shown in Fig. 7(a). A larger displacement leads to larger mechanical energy. The second mechanical energy is larger than the first one for both piezoelectric layers. Fig. 7(c) shows the strain on the nonlinear U-VPEH model, the definition of the strain recorded here is the first principle strain on the piezoelectric layer. Therefore, we can draw the conclusion that for both the first and the second piezoelectric, the second maximum strain is a slightly lower than the first maximum strain because of the large damping on the second vibration mode.

In summary, the nonlinear U-VPEH model can yield closer two frequencies band gap compared to the linear one. The first and second eigenfrequencies are 17.167 Hz and 22.951 Hz for the nonlinear U-VPEH model while the linear model yields 17.366 Hz and 25.124 Hz, respectively. The displacement, the mechanical energy, and the strain were also investigated and these properties of the nonlinear U-VPEH model were all larger than those of the linear model, both for the primary beam and the auxiliary beam.

4. Theoretical analysis

Fig. 8 describes the equivalent lumped nonlinear U-VPEH model. The primary beam can be equivalent to a lumped mass-spring-damping mechanical system with a repelling force generated between the tip mass and fixed permanent magnets, while the auxiliary beam is equivalent to the lumped mass-spring-damping mechanical system. $m_1$ and $m_2$ represent the lumped mass of the primary beam and the auxiliary beam, respectively. $R$ denotes the external resistor load. The nonlinear magnetic potential exhibits a double-well shape. The repelling force can be assumed as [41]:

$$F_m = -a(y_1 - y_0) - b(y_1 - y_0)^3 = \frac{-a(y_1 - y_0) - b(y_1 - y_0)^3}{m_1}$$

(1)
Fig. 7. Finite element results of the piezoelectric layer on the nonlinear U-VPEH model: (a) displacement; (b) mechanical energy; (c) strain.

Fig. 8. The equivalent lumped nonlinear U-VPEH model.
where $F_m$ denotes the magnetic force, $y_1$ denotes the displacement of the primary beam and $y_0$ denotes the displacement of the auxiliary beam. The coefficients of $a$ and $b$ are the empirical parameters that define the repelling force. If $b > 0$, the magnetic potential cause a hardening effect whereas if $b < 0$, it cause a softening effect [41]. The mass matrix of the lumped model can be defined as:

$$[M] = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$  \hspace{1cm} (2)$$

where $m_1$ and $m_2$ represent the lumped mass of the primary beam and auxiliary beam of the vibration-based energy harvester.

The linear stiffness of the two beams are denoted by $k_1$ and $k_2$, respectively. After connecting the two beams, the stiffness matrix of the lumped model is represented as:

$$[K] = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & k_2 \\ -k_2 & k_2 \end{pmatrix}$$  \hspace{1cm} (3)$$

The stiffness matrix in Eq. (3) does not precisely describe the dynamic response mechanism of the two-beam system. It should be mentioned that the stiffness matrix was derived by the inverse of flexibility matrix. According to the research work by Wu [34], the stiffness matrix of the non-continuous structure is modelled as:

$$[K] = \frac{6EI_1}{(4EI_1 \beta^3 + 3EI_2 \beta^2)L_1^3} \begin{pmatrix} 2EI_1 \beta^3 + 2EI_2 + 6EI_1 \beta^2 - 6EI_2 \beta & -2EI_2 + 3EI_2 \beta \\ -2EI_2 + 3EI_2 \beta & 2EI_2 \end{pmatrix}$$  \hspace{1cm} (4)$$

where $E$ is the Young’s modulus; $L_1$ and $L_2$ represent the length of the primary beam and the auxiliary beam, respectively; $\beta = L_2/L_1$ is the length ratio of first beam over the second beam, $I_1$ and $I_2$ denote the cross-section moment of the primary and auxiliary beam, respectively.

The piezoelectric material is attached to the upper surface of the primary beam and the lower surface of the auxiliary beam on the proposed 2-degree-of-freedom structure, respectively. Combining the mechanic part and the electric part together, a nonlinear U-VPEH model is built. The governing motion equation could be obtained:

$$\begin{align*}
    m_1 \ddot{y}_1 + \eta_1 (\dot{y}_1 - \dot{y}_0) + \eta_2 (\dot{y}_1 - \dot{y}_2) + k_{11} (y_1 - y_0) + k_{12} (y_2 - y_0) + \alpha_1 V_1 - \alpha_2 V_2 &= F_m \\
    m_2 \ddot{y}_2 + \eta_2 (\dot{y}_2 - \dot{y}_1) + k_{21} (y_1 - y_0) + k_{22} (y_2 - y_0) + \alpha_2 V_2 &= 0 \\
    V_1/R &= \alpha_1 (y_1 - y_0) - C \ddot{V}_1 \\
    V_2/R &= \alpha_2 (y_2 - y_1) - C \ddot{V}_2
\end{align*}$$  \hspace{1cm} (5)$$

where the displacement of the base is defined as $y_0$ along the direction of excitation and the displacement of the primary beam as $y_1$, the auxiliary beam as $y_2$, so it is obvious that the acceleration of the excitation can be denoted as $\ddot{y}_0$. The dotted terms mean the derivatives of time, $\alpha$ means the electromechanical coupling coefficient and $C$ denotes the clamped capacitance of the piezoelectric element. The electromechanical coupling coefficient is an index that reflects the ability of the piezoelectric system converting mechanical deformation into electrical power.

Giving the following definition:

$$\begin{align*}
    \delta_1 &= \frac{\eta_1}{2m_1} \\
    \delta_2 &= \frac{\eta_2}{2m_2} \\
    \xi_1 &= \frac{k_{11}}{m_1}, \xi_2 = \frac{k_{12}}{m_1}, \xi_3 = \frac{k_{21}}{m_2}, \xi_4 = \frac{k_{22}}{m_2}
\end{align*}$$

Then, Eq. (5) could be rewritten as Eqs. (6)–(9):

$$\begin{align*}
    \ddot{u}_1 + 2\delta_1 (\dot{u}_1 - \dot{y}_0) + 2\delta_2 (\dot{u}_2) + \xi_1 (u_1 - y_0) + \xi_2 (u_1 + u_2 - y_0) + \frac{\alpha_1}{m_1} V_1 - \frac{\alpha_2}{m_1} V_2 &= \frac{F_m}{m_1} \\
    \ddot{u}_1 + \ddot{u}_2 + 2\delta_3 \ddot{u}_2 + \xi_3 (u_1 - y_0) + \xi_4 (u_1 + u_2 - y_0) + \frac{\alpha_2}{m_2} V_2 &= 0 \\
    \frac{V_1}{R} &= \alpha_1 (\dot{u}_1 - \dot{y}_0) - C \ddot{V}_1 \\
    \frac{V_2}{R} &= \alpha_2 \ddot{u}_2 - C \ddot{V}_2
\end{align*}$$  \hspace{1cm} (6)$$

The base excitation is assumed as harmonic vibration, then the dynamic response can be represented as following:
Substituting Eq. (10) into Eq. (8) and Eq. (9), and assuming \( s = j \omega \), then the voltage response \( V_1 \) and \( V_2 \) can be obtained:

\[
\begin{align*}
V_1 &= \frac{j \omega \varphi_2}{j \omega C_1 + \frac{1}{R}} (u_1 - y_0) \\
V_2 &= \frac{j \omega \varphi_2}{j \omega C_1 + \frac{1}{R}} u_2
\end{align*}
\] 

(11)

Substituting Eq. (10) and Eq. (11) into Eq. (7):

\[
(-\omega^2 + \zeta_3 + \zeta_4)u_1 + \left(-\omega^2 + \zeta_4 + 2j \omega \delta_1 + \frac{m_2}{m_2} \frac{j \omega \varphi_2}{j \omega C_1 + \frac{1}{R}} \right) u_2 = (\zeta_3 + \zeta_4) y_0
\] 

(12)

In order to reduce the calculation complexity, combing the Eq. (11) and the Eq. (12), the following equation is obtained:

\[
\begin{align*}
V_1 &= N_0(u_1 - y_0) \\
V_2 &= N_1 u_2 \\
u_2 &= N_2 y_0 + \zeta_3 u_1
\end{align*}
\] 

(13)

Supposing the excitation signal is harmonic and the signal is shown as follows:

\[
\begin{align*}
\tilde{y}_0 &= A_0 e^{j \omega t} \\
u_1 &= A_1 e^{j \omega t + \phi_1} \\
u_2 &= A_2 e^{j \omega t + \phi_2} \\
V_1 &= A_3 e^{j \omega t + \phi_3} \\
V_2 &= A_4 e^{j \omega t + \phi_4}
\end{align*}
\] 

(14)

where \( A_0, A_1, A_2, A_3, A_4 \) stand for the amplitudes of these harmonic responses, \( \phi_1, \phi_2, \phi_3, \phi_4 \) are the phase differences. Therefore, the real part of the two sides of the Eq. (13) are equal to each other, respectively, then:

\[
\begin{align*}
re(V_1) &= re(N_0) re(u_1) - im(N_0) im(u_1) - re(N_0) re(y_0) + im(N_0) im(y_0) \\
re(V_2) &= re(N_1) re(y_0) - im(N_1) im(y_0) + re(N_1) re(u_1) - im(N_1) im(u_1)
\end{align*}
\] 

(15)

where \( re() \) and \( im() \) represent the real part and imaginary part.

\[
\begin{align*}
re(V_1) &= re(N_0) A_1 \cos(\omega t + \phi_1) - im(N_0) A_1 \sin(\omega t + \phi_1) + re(N_0) A_0 \cos \omega t - im(N_0) A_0 \sin \omega t \\
re(V_2) &= -re(N_1) N_1 \cos \omega t + im(N_1) N_1 \cos \omega t + re(N_1) A_1 \cos(\omega t + \phi_1) - im(N_1) N_1 \cos(\omega t + \phi_1)
\end{align*}
\] 

(16)

According to the Euler’s formula, if we drop the imaginary part of the Eq. (14), then the excitation signal can be written in another form shown as following:

\[
\begin{align*}
\tilde{y}_0 &= A_0 \cos(\omega t) \\
u_1 &= A_1 \cos(\omega t + \phi_1) \\
u_2 &= A_2 \cos(\omega t + \phi_2) \\
V_1 &= A_3 \cos(\omega t + \phi_3) \\
V_2 &= A_4 \cos(\omega t + \phi_4)
\end{align*}
\] 

(17)

Substituting the Eq. (16) and the Eq. (17) into the Eq. (6), and neglecting high order harmonic \( \cos(3 \omega t) \), equalizing the terms of \( \cos(\omega t) \) and \( \sin(\omega t) \) separately, the coefficients between the two sides of the equations are equal, then:

\[
N_4 \sin \phi_1 + N_5 \cos \phi_1 = N_6
\] 

(18)

\[
N_4 \cos \phi_1 - N_5 \sin \phi_1 = N_7
\] 

(19)

Squaring both sides of the Eq. (18) and the Eq. (19) and adding them yields:

\[
N_4^2 + N_5^2 = N_6^2 + N_7^2
\] 

(20)

For a given base excitation \( \tilde{y}_0 = A_0 \cos(\omega t) \) to the equation (20), with known \( A_0 \), the amplitude \( A_1 \) of \( u_1 \) can also be solved, then with the help of \( u_1 \) and \( y_0 \), the amplitude \( A_3 \) of \( V_1 \) can be solved.

Similarly, substituting the Eq. (16) and the Eq. (17) into the Eq. (7), the following equation is obtained:
Physical properties of the proposed U-VPEH model.

\[ N^2 + N^2 = N_{10}^2 + N_{11}^2 \]  

Through the Eq. (21), the amplitude \( A_2 \) of \( u_2 \) can be solved, then the amplitude \( A_1 \) of \( V_2 \) also can be obtained with the help of the equation.

Where \( V_1 \) denotes the voltage on the first piezoelectric and \( V_2 \) denotes the voltage generated on the second piezoelectric. According to Ref. [42], the total voltage response can be expressed as:

\[ V_{\text{total}} = V_1 + V_2 \]  

Afterward, the power response is accordingly given by:

\[ P = \frac{V_{\text{total}}^2}{R} \]  

In order to validate the analytical solutions, the governing equations could also be solved by using the Runge-Kutta method. The equations of the Runge-Kutta is derived by a state space matrix. Here, the state space form is defined as:

\[ [y_1 \ y_2 \ y_3 \ y_4 \ y_5] = [u_1 \ u_2 \ \dot{u}_1 \ \dot{u}_2 \ V_1 \ V_2] \]  

Then the Eq. (6) to the Eq. (9) can be reorganized as:

\[
\begin{align*}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= \frac{\left[ \gamma_1 (y_1 - y_0) - \gamma_2 (y_1 + y_2 - y_0) - 2 \delta_1 (y_1 - y_0) + 2 \delta_2 y_4 - \frac{2 \gamma_1}{m_1} y_5 - \frac{2 \gamma_2}{m_2} y_6 - \frac{2 \delta_1}{m_1} (y_1 - y_0) - \frac{2 \delta_2}{m_2} (y_1 - y_0)^3 \right]}{m_1} \\
\dot{y}_4 &= \left( \gamma_1 - \gamma_2 \right) (y_1 - y_0) + \left( \gamma_2 - \gamma_4 \right) (y_1 + y_2 - y_0) + 2 \delta_1 (y_3 - y_0) - 2 \delta_2 (y_3 - y_0)^3 y_4 + \left( \frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_2} \right) y_5 + \left( \frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_2} \right) y_6 + \frac{2 \delta_1}{m_1} (y_1 - y_0) + \frac{2 \delta_2}{m_2} (y_1 - y_0)^3 \\
\dot{y}_5 &= \frac{\gamma_1}{m_1} y_4 - \frac{\gamma_2}{m_2} \\
\dot{y}_6 &= \frac{\gamma_1}{C_1} y_4 - \frac{\gamma_2}{C_2} 
\end{align*}
\]  

The numerical results of the differential Eq. (25) can be got by the Runge-Kutta method in MATLAB. Table 1 shows the physical properties of the proposed U-VPEH model.

### 5. Results and discussions

This section describes the results of theoretical analysis and modeling validation of the proposed both linear and nonlinear U-VPEH model. The voltage response of the linear and nonlinear energy harvester model is given and the corresponding resonance frequency is also provided in time domain analysis. For the nonlinear voltage response, the excitation swept signal is defined under both up-sweeping and down-sweeping in order to observe the performance of the nonlinear energy harvester. Here, the primary beam and the auxiliary beam are expressed with the 1st beam and the 2nd beam in order to simplify the description. The definition of the total beam is the summation of the primary (1st) beam and the auxiliary (2nd) beam.

Fig. 9 shows voltage response of the linear U-VPEH model in the swept signals from 12 Hz to 26 Hz. By comparing of Fig. 9 (a) and (b), one conclusion can be drawn which is the voltage response of the primary beam is much smaller than that of the auxiliary beam. The piezoelectric on the auxiliary beam dominated the main power output. For the voltage response on the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>Length of the primary beam</td>
<td>0.1</td>
<td>m</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>Length of the auxiliary beam</td>
<td>0.0655</td>
<td>m</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>Width of the primary beam</td>
<td>0.04</td>
<td>m</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>Width of the auxiliary beam</td>
<td>0.017</td>
<td>m</td>
</tr>
<tr>
<td>( h )</td>
<td>Thickness of the primary and auxiliary beam</td>
<td>0.006</td>
<td>m</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>Lumped mass of the primary beam</td>
<td>7.2</td>
<td>g</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>Lumped mass of the auxiliary beam</td>
<td>10.8</td>
<td>g</td>
</tr>
<tr>
<td>( E )</td>
<td>Yong modulus of aluminum beam</td>
<td>71</td>
<td>GPa</td>
</tr>
<tr>
<td>( I_1 )</td>
<td>Moment of inertia of the primary beam</td>
<td>( 7.2 \times 10^{-11} )</td>
<td>kg m²</td>
</tr>
<tr>
<td>( I_2 )</td>
<td>Moment of inertia of the auxiliary beam</td>
<td>( 3.06 \times 10^{-13} )</td>
<td>kg m²</td>
</tr>
<tr>
<td>( a )</td>
<td>Coefficient of the magnetic force</td>
<td>(-0.56 \times 10^2 )</td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td>Coefficient of the magnetic force</td>
<td>(-6.6 \times 10^6 )</td>
<td></td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>The electromechanical coupling coefficient of piezoelectric at the primary beam</td>
<td>( 4.59 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>The electromechanical coupling coefficient of piezoelectric at the auxiliary beam</td>
<td>( 1.125 \times 10^{-4} )</td>
<td></td>
</tr>
<tr>
<td>( C_1 )</td>
<td>The clamped capacitance of the piezoelectric element</td>
<td>( 2.57 \times 10^{-8} )</td>
<td></td>
</tr>
</tbody>
</table>
auxiliary beam, the voltage on the first resonance frequency outperforms that on the second resonance frequency. In one word, the maximum voltage response can be achieved on the auxiliary beam at its first resonance vibration mode under a swept signal excitation. Fig. 9(c) displays the total beams’ voltage response during the swept signals from 12 Hz to 26 Hz. The resonance frequencies of the total beams are 16.56 Hz and 23.13 Hz, which is very close to the simulation resonance frequencies shown in Fig. 2. The total voltages of the two beams are 6.699 V @16.56 Hz and 1.338 V @23.13 Hz. Obviously, the first voltage response is much larger than the second voltage response for the total beams.

Fig. 9. Voltage response of the linear U-VPEH model in swept signals from 12 Hz to 26 Hz: (a) the 1st beam; (b) the 2nd beam; (c) the total beam.

Fig. 10. Voltage response of the nonlinear U-VPEH model in upwards swept signals from 12 Hz to 26 Hz: (a) the 1st beam; (b) the 2nd beam; (c) the total beam.
In order to eliminate the nonlinear response delay error, the nonlinear energy harvester is excited by the upwards and downwards swept sine signal. Figs. 10 and 11 show voltage response of the proposed nonlinear U-VPEH model in upwards and downwards signals, respectively. The eigenfrequency is 16.81 Hz for the 1st beam and 16.44 Hz for the 2nd beam under the upwards swept signals. There is a little difference for the voltage response under the downwards swept signals, the eigenfrequency is 15.43 Hz for the 1st beam and 15.81 Hz for the 2nd beam as described in Fig. 11. The first up-swept eigenfrequency for the whole system is 16.5 Hz shown in Fig. 10(c) and the first down-swept eigenfrequency is 15.69 Hz shown in Fig. 11(c).

Fig. 12 shows the linear voltage response of the proposed energy harvester under the base excitation of 0.1 g with a resistor of 10 MΩ. The linear voltage response of the proposed energy harvester is obtained if the values of the parameter \(a\) and \(b\) in Eq. (1) are set up to zero. The voltage on the 1st beam is much lower than that of the 2nd beam from the results as shown in Fig. 12(a) and (b), no matter for the first resonance frequency or the second frequency.

For the first resonance frequency, the linear voltage response is 0.6245 V@16.2 Hz for the 1st beam while the linear voltage response is 3.984 V@16 Hz for the 2nd beam. For the second resonance frequency, the voltage response is 0.5713 V@22.6 Hz for the 1st beam while the voltage response is 2.764 V@23 Hz for the 2nd beam.

Therefore, if the proposed U-VPEH model performed the linearity, then the primary beam contributes a very small voltage response to the whole structure power output and the auxiliary beam dominated the voltage response and the power output. Additionally, the timing point of the second resonance frequency is far away from the first resonance frequency and this cannot keep a continuous and stable high power output. Lastly, the two voltage responses for the whole structure at the both resonance frequencies are similar. These phenomena are the weaknesses of the linear U-VPEH model. Instead of shifting these two resonance frequencies moves close to each other, a wide bandwidth of the frequency range is proposed which can cover the first and the second resonance frequency through the exploitation of the nonlinearity to the U-VPEH model.

Fig. 13 shows the nonlinear voltage response of the proposed U-VPEH model under the base excitation of 0.1 g with a resistor of 10 MΩ, which the Fig. 13(a) is the 1st beam, Fig. 13(b) is the 2nd beam and Fig. 13(c) is the total beams. Due to the coefficient \(b\) in the repelling force Eq. (1) is negative, then it caused a softening effect on this nonlinear energy harvester. The softening effect can make the resonance frequency moves from the relatively high frequency to lower frequency and broaden the bandwidth of the resonance vibration range. Obviously, the 2nd resonance frequency moves to the 1st resonance frequency and combined a wide frequency range for the auxiliary beam response.

Fig. 14 describes the performance comparison of the linear voltage response between the analytical solution and time domain solution, respectively. As illustrated in Fig. 14, the analytical response fits the time domain response very well. The first resonance in time domain response is 16.53 Hz with the amplitude of 6.703 V in while the first resonance of the analytical response is 16 Hz with the amplitude of 4.608 V. The second resonance frequency is 23.19 Hz with the amplitude of 2.869 V in time domain response while the second resonance of the analytical response is 23.2 Hz with the amplitude of 3.253 V.

![Fig. 11. Voltage response of the nonlinear U-VPEH model in downwards swept signals from 26 Hz to 12 Hz: (a) the 1st beam; (b) the 2nd beam; (c) the total beam.](image-url)
**Fig. 12.** The linear voltage response of the proposed U-VPEH model under the base excitation of 0.1 g with a resistor of 10 MΩ: (a) the 1st beam; (b) the 2nd beam; (c) the total beam.

**Fig. 13.** The nonlinear voltage response of the proposed U-VPEH model under the base excitation of 0.1 g with a resistor of 10 MΩ: (a) the 1st beam; (b) the 2nd beam; (c) the total beam.
Fig. 14. Comparison of the linear voltage response between the analytical solution and time domain solution.

Fig. 15. Comparison of the nonlinear voltage response between the analytical solution and time domain solution.
The error between the time domain solution and the analytical solution is very minor and acceptable. Therefore, the method of the analytical solution has been demonstrated is right and reasonable in the linear energy harvester. However, this cannot be taken as an evidence that the approach of the analytical analysis can be used to describe the nonlinear energy harvester.

Fig. 15 shows the nonlinear voltage response between the analytical solution and the time domain solution. The blue line represents the upwards response and the brown line means the downwards response. The magenta dot is the result of the analytical response. From this figure, we can see that the nonlinear response is not obvious when the excitation signal sweeps from low frequency 12 Hz to high frequency 26 Hz, however, the down-sweep response under the frequency range from 26 Hz to 12 Hz matches very well with the analytical response. The maximum voltage response is 8.743 V@16.5 Hz under the up-sweeping signals while the maximum voltage response is 14.18 V@15.41 Hz under the down-sweeping signals. The voltage response for the analytical solution is 9.195 V@14.8 Hz.

6. Experimental study under swept sine excitation

In order to validate the results from the analytical and time domain solution are reliable and the proposed model has the potential to be exploited as an energy harvesting device, here an experimental platform is built up. The model is fabricated according to the physical properties shown in Table 1. The configuration is fabricated and tested by the piezoelectric material, the aluminum beam, and the rectangular permanent magnets. The type of the piezoelectric material is PPA-2014 package and purchased from the MIDE Company. The Length \( \times \) Width \( \times \) Thickness of the PPA-2014 is 53 mm \( \times \) 20.8 mm \( \times \) 0.83 mm.

Fig. 16 shows the schematic of the experimental setup which is used to validate the results generated from the numerical analysis. As illustrated, the experimental platform consists of the Function Generator, the Pulse Labshop, the Power Amplifier, the external resistor box and the B&K shaker. The excitation frequency is generated by the Function Generator and the sweeping sinusoidal signal range is set up to from 0 Hz to 30 Hz. The voltage response and resonance frequency are
measured by the Pulse Labshop software which is installed on a computer. The other factors, such as the masses of screws, tapes, are neglected. The twist of the beams is very tiny and is neglected. The external resistor is set up to 10 MΩ in consistent with the resistor conducted in the simulation study.

Figs. 17 and 18 demonstrated the experimental voltage response for both the linear and the nonlinear U-VPEH system. It can be seen from Fig. 17(c), the experimental voltage response for the total beams on the linear U-VPEH is 10.65 V at the first resonance frequency of 16.02 Hz and 7.224 V at the second resonance frequency of 18.72 Hz. The resonance frequency has a
little difference with the analytical and time domain analysis shown in Fig. 14. Besides, the voltage generated on the 2nd beam contributed more than that of the 1st beam to the whole experimental voltage response which is in consistent with the results expressed in Fig. 14.

From Fig. 18, the experimental voltage response for the total beams on the nonlinear U-VPEH is 4.612 V at the frequency of 12.58 Hz under the downwards swept sine signals. In comparison to the linear U-VPEH system, the frequency with maximum voltage output has been reduced nearly 3.5 Hz while the peak of the voltage response also decreased. Given an upwards swept sine signal, the maximum power output peak reaches later than the downwards swept sine. The maximum power output for the upwards and downwards keeps the same as 4.5 V during the designed frequency range. When compares with the simulation results shown in Figs. 10 and 11, the eigenfrequencies from the experimental results have a slight reduction, but this still is under the acceptable frequency range. The amplitude of the voltage output of the experimental study has a significant half decrease than the simulation results. This may be the results of the internal and external damping, and also the fabrication process of the model is not completely consistent with the simulated model. Nevertheless, one meaningful point of this proposed structure can absorb the wasted and destructive vibration and reduce the abrasion of the mechanical equipment.

7. Summary

This reported research work introduces a horizontal asymmetric U-shaped vibration-based piezoelectric energy harvester (U-VPEH) model, which consists of the geometric and magnetic nonlinearity coupling with multimodality. The corresponding finite element analysis and the theoretical analysis are conducted. The results indicate that this proposed nonlinear U-VPEH model yields a closer two resonance frequencies band gap compared to the linear one. The first and the second eigenfrequency is 17.167 Hz and 22.951 Hz for the nonlinear U-VPEH model while the linear model is 17.366 Hz and 25.124 Hz, respectively. The primary beam and the auxiliary beam of the nonlinear U-VPEH model achieve a better mechanical performance than the linear one, including the displacement, the mechanical energy, and the strain. The nonlinear U-VPEH model can produce a higher voltage output than the linear counterpart. The voltage response generated by the piezoelectric material on the auxiliary beam dominates the most power output.

The nonlinear U-VPEH model yields low frequencies bandwidth which can cover the maximum power output frequency range from the theoretical analysis. The maximum voltage response is 8.743 V @16.5 Hz under the up-sweeping signals while the maximum voltage response is 14.18 V @15.41 Hz under the down-sweeping signals. The voltage response for the analytical solution is 9.195 V @14.8 Hz. The experimental results demonstrate that the voltage response and the resonance frequency of the U-VPEH are in agreement with the analytical and theoretical analysis.

Overall, the nonlinear horizontal asymmetric U-VPEH model exhibits a promising performance on the energy transfer and power output. This proposed structure can absorb the wasted and destructive vibration and reduce the abrasion of the mechanical equipment. This property of higher energy output, lower resonance frequency, and closer resonance peak can broaden the flexibility and its practical usage.

Acknowledgement

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Appendix A.

\[
N_0 = \frac{j \omega \xi_1}{j \omega C_s + \frac{1}{R}}
\]

\[
N_1 = \frac{j \omega \xi_2}{j \omega C_s + \frac{1}{R}}
\]

\[
N_2 = -\frac{\xi_3 + \xi_4}{-\omega^2 + \xi_4 + 2j \omega \delta_3 + \frac{2e_j \xi_4}{m_j \rho_j \omega^2 + \frac{1}{R}}}
\]

\[
N_3 = -\frac{-\omega^2 + \xi_3 + \xi_4}{-\omega^2 + \xi_4 + 2 \delta_3 \omega j + \frac{2e_j \xi_4}{m_j \rho_j \omega^2 + \frac{1}{R}}}
\]

\[
N_4 = -\frac{-\omega^2 + \xi_3 + \xi_4}{-\omega^2 + \xi_4 + 2 \delta_3 \omega j + \frac{2e_j \xi_4}{m_j \rho_j \omega^2 + \frac{1}{R}}}
\]
N_5 = -A_1 \omega^2 + (\zeta_1 + \zeta_2) A_1 + \zeta_2 \text{re}(N_3) A_1 + 2\delta_2 \text{im}(N_3) \omega A_1 + \frac{2i}{m_1} \text{re}(N_1 N_3) A_1 + \frac{2i}{m_1} \text{re}(N_0) A_1 + \frac{4}{m_1} A_1 + \frac{4}{m_1} A_1^2$

N_6 = -\left( (\zeta_1 + \zeta_2) \frac{A_0}{\omega^2} + \zeta_2 \text{re}(N_2) A_0 + 2\delta_2 \text{im}(N_2) \omega A_0 + \frac{2i}{m_1} \text{re}(N_0) \frac{A_0}{\omega^2} + \frac{2i}{m_1} \text{re}(N_1 N_2) A_0 - \left( \frac{A_0}{m_1} + \frac{4}{m_1} A_1 \right) \text{re}(N_0) A_0 - \frac{A_0}{m_1} \text{im}(N_0) A_0 \right)\text{re}(N_0) A_0$

N_7 = \zeta_3 \text{im}(N_2) A_0 + 2\delta_1 A_0 - \frac{2\delta_2 \text{re}(N_2) \omega A_0 + \frac{2i}{m_1} \text{im}(N_1 N_2) A_0 + \frac{A_1}{m_1} \text{im}(N_0) A_0}{\omega^2}$

N_8 = \text{im}\left( \frac{A_0}{\omega^2} \right) \omega^2 A_2 - (\zeta_1 + \zeta_2) \frac{A_0}{\omega^2} A_2 - 2\delta_1 \text{re}\left( \frac{A_0}{\omega^2} \right) A_0 + \frac{2i}{m_1} \text{re}(N_1 N_2) A_2 + \frac{A_1}{m_1} \text{im}(N_0) A_2 + \frac{4}{m_1} A_0 + \frac{4}{m_1} A_1 \text{re}(N_0) A_2$

N_9 = -\left( \frac{A_0}{\omega^2} \right) \omega^2 A_2 + (\zeta_1 + \zeta_2) \frac{A_0}{\omega^2} A_2 - 2\delta_1 \text{re}(N_0) \frac{A_0}{\omega^2} + \frac{2i}{m_1} \frac{1}{\omega^2} \text{re}(N_1 N_2) A_2 + \left( \frac{A_1}{m_1} + \frac{4}{m_1} A_1 \right) \frac{1}{\omega^2} \text{re}(N_0) A_2$

N_{10} = -\left( \frac{A_0}{\omega^2} \right) \omega^2 A_2 - (\zeta_1 + \zeta_2) \frac{A_0}{\omega^2} A_2 + \frac{2i}{m_1} \frac{1}{\omega^2} \text{re}(N_1 N_2) A_2 - \frac{A_1}{m_1} \frac{1}{\omega^2} \text{re}(N_0) A_2 - \left( \frac{A_1}{m_1} + \frac{4}{m_1} A_1 \right) \frac{1}{\omega^2} \text{re}(N_0) A_2$

N_{11} = -\left( \frac{A_0}{\omega^2} \right) \omega^2 A_2 + \frac{2i}{m_1} \frac{1}{\omega^2} \text{re}(N_1 N_2) A_2 - 2\delta_1 \text{re}(N_0) \frac{A_0}{\omega^2} A_2 + \frac{2i}{m_1} \frac{1}{\omega^2} \text{re}(N_1 N_2) A_2 - \left( \frac{A_1}{m_1} + \frac{4}{m_1} A_1 \right) \frac{1}{\omega^2} \text{re}(N_0) A_2$


