

ON THE CHARACTERIZATIONS OF MATRIX FIELDS AS LINEARIZED STRAIN TENSOR FIELDS

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Abstract

Saint Venant's and Donati's theorems constitute two classical characterizations of smooth matrix fields as linearized strain tensor fields. Donati's characterization has been extended to matrix fields with components in L^2 by T. W. Ting in 1974 and by J. J. Moreau in 1979, and Saint Venant's characterization has been extended likewise by the second author and P. Ciarlet, Jr. in 2005. The first objective of this paper is to further extend both characterizations, notably to matrix fields whose components are only in H^{-1} , by means of different, and to a large extent simpler and more natural, proofs. The second objective is to show that some of our extensions of Donati's theorem allow to reformulate in a novel fashion the pure traction and pure displacement problems of linearized three-dimensional elasticity as quadratic minimization problems with the strains as the primary unknowns. The third objective is to demonstrate that, when properly interpreted, such characterizations are "matrix analogs" of well-known characterizations of vector fields. In particular, it is shown that Saint Venant's theorem is in fact nothing but the matrix analog of Poincaré's lemma.

Keywords. Saint Venant compatibility conditions; Donati's theorem; Korn's inequality; Poincaré's lemma; linearized elasticity

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Titre: Sur les caractérisations des champs de matrices comme des champs de tenseurs de déformation linéarisés

Résumé: Les théorèmes de Saint Venant et de Donati constituent deux caractérisations classiques de champs de matrices réguliers comme des champs de tenseurs de déformation linéarisés. La caractérisation de Donati a été étendue aux champs de matrices dont les composantes sont dans L^2 par T.W. Ting en 1974 et par J.J. Moreau en 1979. La caractérisation de Saint Venant a été pareillement étendue par le second auteur et P. Ciarlet, Jr. en 2005. Le premier objectif de cet article est de montrer que l'on peut généraliser encore davantage ces caractérisations, en particulier à des champs de matrices dont les composantes sont seulement dans H^{-1} , au moyen de démonstrations différentes, et dans une large mesure plus simples et plus naturelles. Le second objectif est de montrer que certaines de nos généralisations du théorème de Donati conduisent à de nouvelles façons de poser les problèmes de traction pure et de déplacement pur de l'élasticité linéarisée tridimensionnelle, sous la forme de problèmes de minimisation quadratique où les déformations deviennent les inconnues principales. Le troisième objectif est de montrer que, une fois convenablement interprétées, ces caractérisations apparaissent comme les "analogues matriciels" de caractérisations bien connues de champs de vecteurs. En particulier, on montre que le théorème de Saint Venant n'est autre que l'analogue matriciel du lemme de Poincaré.

1 Introduction

Before describing the content of this paper, we first briefly review the genesis of the classical characterizations of matrix fields as linearized strain tensor fields, as well as their various subsequent extensions (for more historical details until 1972, see Gurtin [18]). The notations used, but not defined, in this introduction are defined in Section 2.

Let Ω be an open subset of \mathbb{R}^3 and let $\mathbf{v} = (v_i)$ be a smooth enough vector field defined over Ω . The symmetric matrix field $\nabla_s \mathbf{v}$ defined over Ω by

$$(\nabla_s \mathbf{v})_{ij} := \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

is the *linearized strain tensor field* associated with the vector field \mathbf{v} .

As is well known, the field $\nabla_s \mathbf{v}$ plays a key role in *linearized three-dimensional elasticity*, where the field \mathbf{v} is interpreted as a displacement field of the set Ω , itself viewed as the reference configuration of a linearly elastic body.

The question of characterizing those symmetric matrix field $\mathbf{e} = (e_{ij})$ that can be written over Ω as

$$\mathbf{e} = \nabla_s \mathbf{v} \text{ for some vector field } \mathbf{v},$$

has been arousing considerable interest for quite a long time. Indeed A. J. C. B. de Saint Venant announced as early as 1864 what is since then known as *Saint Venant's theorem* (in fact, it was not until 1886 that E. Beltrami provided a rigorous proof of this result): *Assume that the open set Ω is simply-connected. Then there exists a vector field $\mathbf{v} \in \mathbf{C}^3(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in Ω if (and clearly only if, even if Ω is not simply-connected) the functions e_{ij} are in the space $C^2(\Omega)$ and they satisfy*

$$R_{ijkl}(\mathbf{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } \Omega \text{ for all } i, j, k, l \in \{1, 2, 3\}.$$

It is easily seen (Theorem 5.1) that the *Saint Venant compatibility conditions* $R_{ijkl}(\mathbf{e}) = 0$ in Ω are equivalent to the relations

$$\mathbf{CURLCURL} \mathbf{e} = \mathbf{0} \text{ in } \Omega;$$

besides the matrix field $\mathbf{CURLCURL} \mathbf{e}$ is always symmetric. Hence the eighty-one relations $R_{ijkl}(\mathbf{e}) = 0$ reduce in effect to six relations only.

It is only recently that the Saint Venant compatibility conditions were shown to remain sufficient under substantially weaker regularity assumptions. More specifically, Ciarlet & Ciarlet, Jr. [10] just established the following *Saint Venant theorem in $\mathbb{L}_s^2(\Omega)$* , where $\mathbb{L}_s^2(\Omega)$ stands for the space of all symmetric matrix fields with components in $L^2(\Omega)$: *Let Ω be a bounded and simply-connected open subset of \mathbb{R}^3 with a Lipschitz-continuous boundary and let $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ be a matrix field that satisfies the Saint Venant compatibility conditions $R_{ijkl}(\mathbf{e}) = 0$ in $H^{-2}(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$.*

Not only is such an extension interesting *per se*, but, perhaps more importantly, it also allows to reformulate in a novel way some classical problems of linearized three-dimensional

elasticity. Indeed, this was the main motivation of Ciarlet & Ciarlet, Jr. [10], who used the Saint Venant theorem in $\mathbb{L}_s^2(\Omega)$ to revisit the *pure traction problem of linearized elasticity*. While the unknown *displacement field* for such a problem is sought as the minimizer in $\mathbf{H}^1(\Omega)$ of a quadratic functional, it is now the *linearized strain tensor field* that is considered as the primary unknown in their new approach. As expected, this new unknown now satisfies a *constrained minimization problem*, in the sense that it minimizes a quadratic functional over the closed subspace of $\mathbb{L}_s^2(\Omega)$ that consists of all matrix fields $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ satisfying the relations $R_{ijkl}(\mathbf{e}) = 0$ in $H^{-2}(\Omega)$. Note in passing that, since the constitutive laws of linearized elasticity are invertible, this constrained minimization problem can be immediately recast as one with the *stresses* as the sole unknowns. This observation paves the way for potentially attractive *finite element methods* (see [11]).

In 1890, L. Donati proved that, *if Ω is an open subset of \mathbb{R}^3 and the components e_{ij} of a symmetric matrix field $\mathbf{e} = (e_{ij})$ are in the space $C^2(\Omega)$ and they satisfy:*

$$\int_{\Omega} e_{ij} s_{ij} dx = 0 \text{ for all } \mathbf{s} = (s_{ij}) \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \Omega,$$

where $\mathbb{D}_s(\Omega)$ denotes the space of all symmetric tensor fields whose components are infinitely differentiable in Ω and have compact supports in Ω , *then*

$$\mathbf{CURLCURL} \mathbf{e} = \mathbf{0} \text{ in } \Omega.$$

This result, known as *Donati's theorem*, thus provides, once combined with Saint Venant's theorem, another characterization of symmetric matrix fields as linearized strain tensor fields, at least for simply-connected open subsets Ω of \mathbb{R}^3 .

A first extension of Donati's theorem was given in 1974 by Ting [27]: *If Ω is bounded and has a Lipschitz-continuous boundary and if the components e_{ij} of the symmetric tensor field \mathbf{e} are in $L^2(\Omega)$ and again satisfy*

$$\int_{\Omega} e_{ij} s_{ij} dx = 0 \text{ for all } \mathbf{s} = (s_{ij}) \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \Omega,$$

then there exists $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$.

Then Moreau [21] showed in 1979 that Donati's theorem holds even in the sense of distributions, according to the following theorem, where Ω is now an arbitrary open subset of \mathbb{R}^3 : *If the components e_{ij} of the symmetric tensor field \mathbf{e} are in $D'(\Omega)$ and satisfy*

$$D'(\Omega) \langle e_{ij}, s_{ij} \rangle_{D(\Omega)} = 0 \text{ for all } \mathbf{s} = (s_{ij}) \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \Omega,$$

then there exists a vector field $\mathbf{v} = (v_i)$ with $v_i \in D'(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in the sense of distributions. Note that Ting's and Moreau's extensions *do not* require that Ω be simply-connected.

The *main objective* of this paper is to provide *further extensions of Donati's and Saint Venant theorems* that hold under a "very weak" regularity assumption on the matrix field \mathbf{e} .

More specifically, we first prove in Section 4 *three different extensions of Donati's theorems*. The first characterization holds for symmetric matrix fields $\mathbf{e} = (e_{ij})$ whose components e_{ij}

are only in $H^{-1}(\Omega)$ (Theorem 4.1). The second and third ones, which both hold for fields \mathbf{e} with components in $L^2(\Omega)$, differ in that the resulting vector field \mathbf{v} (i.e., the field that satisfies $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$) is found either in $\mathbf{H}_0^1(\Omega)$ (Theorem 4.2) or in $\mathbf{H}^1(\Omega)$ (Theorem 4.3). Note that analogous results have been simultaneously and independently obtained by Geymonat & Krasucki [14], albeit by different methods.

We then show in Section 5 how these extensions of Donati’s theorem allow to reformulate in a novel way the pure traction problem (Theorem 5.1) and the pure displacement problem (Theorem 5.3) of linearized three-dimensional elasticity, as constrained quadratic minimization problems with the *linearized strain tensor as the primary unknown*. This approach, which is called “intrinsic” in the Engineering and Computational Mechanics circles (see, e.g., Opoka & Pietraszkiewicz [24]), presents the advantage of directly providing the *stress tensor field*, since the constitutive equation of a linearly elastic material is invertible. Note also that such a reformulation also provides a *new proof of the classical Korn inequality* (Theorem 5.2).

Finally, we prove in Section 7 an *extension of Saint Venant’s theorem* that holds if the components of the symmetric matrix field \mathbf{e} are only in $H^{-1}(\Omega)$ (Theorem 7.1), in which case the vector field \mathbf{v} is of course only in $\mathbf{L}^2(\Omega)$. Not surprisingly, we also recover as a corollary (Theorem 7.2) the Saint Venant theorem in $\mathbb{L}_s^2(\Omega)$ (mentioned earlier) of Ciarlet & Ciarlet, Jr. [10], albeit with a substantially simpler proof.

The keystone of our analysis is a *matrix analog of the lemma of J. L. Lions* (Theorem 3.1). Other key ingredients (used in the proof of Theorem 7.1) are a *matrix analog of the well-known Stokes problem*, the *hypocoellipticity of the Laplacian*, and the “classical” *Saint Venant theorem*. Otherwise, the novelty of our approach is reminiscent of that used by Kesavan [19], who provided an illuminating proof of *Poincaré’s lemma*, based on a generalization due to Amrouche & Girault [4] of a well-known *lemma of J. L. Lions* (precise statements of this lemma and of its generalization are found in the proof of Theorem 3.1).

Another objective of this paper is to show that, when put in a proper perspective, *the above extensions appear as natural “matrix analogs” of well-known characterizations of vector fields* (in particular because the “matrix” operators ∇_s and $\mathbf{CURLCURL}$ are, in some respects at least, the matrix analogs of the “vector” operators \mathbf{grad} and \mathbf{curl} ; see Sections 3 and 5). In this respect, a worthwhile conclusion of the present study, discussed at the end of Section 7, is that *our extension of Saint Venant’s theorem is nothing but the matrix analog of a weak form of Poincaré’s lemma*.

The results of this paper were announced in [2] and [3].

2 Notations and preliminaries

Throughout this article, Latin indices vary in the set $\{1, 2, 3\}$ save when they are used for indexing sequences, and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

All the vector spaces considered in this article are over \mathbb{R} . Let V denote a normed vector space with norm $\|\cdot\|_V$. Given a closed subspace Z of V , the equivalence class of $v \in V$ in the

quotient space $\dot{V} := V/Z$ is denoted \dot{v} and its norm is defined by $\|\dot{v}\|_{\dot{V}} := \inf_{z \in Z} \|v + z\|_V$. The notation V' designates the dual space of V and $\langle \cdot, \cdot \rangle_V$ denotes the duality bracket between V' and V . Given a subspace W of V , the notation $W^0 := \{v' \in V'; \langle v', w \rangle_V = 0 \text{ for all } w \in W\}$ designates the polar set of W .

Let U and V denote two vector spaces and let $A : U \rightarrow V$ be a linear operator. Then $\mathbf{Ker} A \subset U$ and $\mathbf{Im} A \subset V$ respectively designate the kernel and the image of A .

Let Ω be an open subset of \mathbb{R}^3 and let $x = (x_i)$ designate a generic point in Ω . Partial derivative operators of the first, second, and third order are then denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$. The same symbols also designate partial derivatives in the sense of distributions.

Spaces of functions, vector fields, and matrix fields, defined over Ω are respectively denoted by italic capitals, boldface Roman capital, and special Roman capitals. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields.

The notations $C^m(\Omega)$, $m \geq 0$, and $C^\infty(\Omega)$ denote the usual spaces of continuously differentiable functions; the notation $D(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and have compact supports in Ω . The notation $D'(\Omega)$ denotes the space of distributions defined over Ω . The notations $H^m(\Omega)$, $m \in \mathbb{Z}$, with $H^0(\Omega) = L^2(\Omega)$, and $H_0^1(\Omega)$ designate the usual Sobolev spaces.

Combining the above rules, we shall thus encounter spaces such as $D'(\Omega)$, $\mathbf{D}'(\Omega)$, $\mathbb{D}'(\Omega)$, $\mathbb{H}_{0,s}^1(\Omega)$, $\mathbb{H}_s^{-1}(\Omega)$, etc. Note that the same notation $\|\cdot\|_{m,\Omega}$, $m \in \mathbb{Z}$, will be used to designate the usual norms in the spaces $H^m(\Omega)$, $\mathbf{H}^m(\Omega)$, and $\mathbb{H}_s^m(\Omega)$.

The notation $(\mathbf{v})_i$ designates the i -th component of a vector $\mathbf{v} \in \mathbb{R}^3$ and the notation $\mathbf{v} = (v_i)$ means that $v_i = (\mathbf{v})_i$. The notation $(\mathbf{A})_{ij}$ designates the element at the i -th row and j -th column of a square matrix \mathbf{A} of order three and the notation $\mathbf{A} = (a_{ij})$ means that $a_{ij} = (\mathbf{A})_{ij}$. The inner-product and vector product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ are denoted $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$. The notation $\mathbf{s} : \mathbf{t} := s_{ij} t_{ij}$ designates the matrix inner-product of two matrices $\mathbf{s} := (s_{ij})$ and $\mathbf{t} := (t_{ij})$ of order three.

The orientation tensor (ε_{ijk}) is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{i, j, k\} \text{ is an even permutation of } \{1, 2, 3\}, \\ -1 & \text{if } \{i, j, k\} \text{ is an odd permutation of } \{1, 2, 3\}, \\ 0 & \text{if at least two indices are equal.} \end{cases}$$

The following differential operators will be constantly used throughout the article: The *vector gradient operator* $\mathbf{grad} : D'(\Omega) \rightarrow \mathbf{D}'(\Omega)$ is defined by

$$(\mathbf{grad} v)_i := \partial_i v \text{ for any } v \in D'(\Omega).$$

The *divergence operator* $\text{div} : \mathbf{D}'(\Omega) \rightarrow D'(\Omega)$ is defined by

$$\text{div } \mathbf{v} :=: \partial_i v_i \text{ for any } \mathbf{v} = (v_i) \in \mathbf{D}'(\Omega).$$

The *vector curl operator* $\mathbf{curl} : \mathbf{D}'(\Omega) \rightarrow \mathbf{D}'(\Omega)$ is defined by

$$(\mathbf{curl} \mathbf{v})_i :=: \varepsilon_{ijk} \partial_j v_k \text{ for any } \mathbf{v} = (v_i) \in \mathbf{D}'(\Omega).$$

The *matrix gradient operator* $\nabla : \mathbf{D}'(\Omega) \rightarrow \mathbb{D}'(\Omega)$ is defined by

$$(\nabla \mathbf{v})_{ij} := \partial_j v_i \text{ for any } \mathbf{v} = (v_i) \in \mathbf{D}'(\Omega).$$

The *vector divergence operator* $\mathbf{div} : \mathbb{D}'(\Omega) \rightarrow \mathbf{D}'(\Omega)$ is defined by

$$(\mathbf{div} \mathbf{e})_i := \partial_j e_{ij} \text{ for any } \mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

The *matrix Laplacian* $\Delta : \mathbb{D}'(\Omega) \rightarrow \mathbb{D}'(\Omega)$ is defined by

$$(\Delta \mathbf{e})_{ij} := \Delta e_{ij} \text{ for any } \mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

The *matrix curl operator* $\mathbf{CURL} : \mathbb{D}'(\Omega) \rightarrow \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{CURL} \mathbf{e})_{ij} := \varepsilon_{ilk} \partial_l e_{jk} \text{ for any } \mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

Finally, a *domain* in \mathbb{R}^3 is a bounded, connected, open subset of \mathbb{R}^3 whose boundary is Lipschitz-continuous in the sense of Nečas [22] or Adams [1].

3 The operator ∇_s

Let Ω be an open subset in \mathbb{R}^3 . For any vector field $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, the symmetric matrix field $\nabla_s \mathbf{v} \in \mathbb{D}'_s(\Omega)$ is defined by

$$\nabla_s \mathbf{v} := \frac{1}{2}(\nabla \mathbf{v}^T + \nabla \mathbf{v}),$$

or equivalently, by

$$(\nabla_s \mathbf{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

When Ω is connected, the kernel of the operator ∇_s has the well-known characterization (see, e.g., [10, Theorem 6.3-4]), viz.,

$$\mathbf{Ker} \nabla_s = \{\mathbf{v} \in \mathbf{D}'(\Omega); \nabla_s \mathbf{v} = \mathbf{0} \text{ in } \mathbb{D}'(\Omega)\} = \{\mathbf{v} = \mathbf{a} + \mathbf{b} \wedge \mathbf{id}_\Omega; \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3\},$$

where \mathbf{id}_Ω denotes the identity mapping of the set Ω .

One objective in this paper is to illustrate why the operator $\nabla_s : \mathbf{D}'(\Omega) \rightarrow \mathbb{D}'_s(\Omega)$ thus defined may be viewed as the “matrix analog” of the “vector” operator $\mathbf{grad} : \mathbf{D}'(\Omega) \rightarrow \mathbf{D}'(\Omega)$. In this direction, a first important property of the operator ∇_s is given in the next theorem. For reasons that will become clear from its proof, this result will be subsequently referred to as the $\mathbb{H}_s^m(\Omega)$ - *matrix version of J. L. Lions’ lemma*.

THEOREM 3.1. *Let Ω be a domain in \mathbb{R}^3 and let a vector field $\mathbf{v} \in \mathbf{D}'(\Omega)$ be such that $\nabla_s \mathbf{v} \in \mathbb{H}_s^m(\Omega)$ for some integer $m \in \mathbb{Z}$. Then $\mathbf{v} \in \mathbf{H}^{m+1}(\Omega)$.*

Proof. A well-known *lemma of J. L. Lions* (first mentioned in 1958; see footnote ⁽²⁷⁾ in Magenes & Stampacchia [20]) asserts the following: Let Ω be a bounded open subset of \mathbb{R}^3 with a smooth boundary and let a distribution $v \in H^{-1}(\Omega)$ be such that $\mathbf{grad} v \in \mathbf{H}^{-1}(\Omega)$. Then $v \in L^2(\Omega)$. After its first published proof appeared in Duvaut & Lions [14, Chapter 3], various extensions of this lemma to Lipschitz-continuous boundaries have been given, notably by Bolley & Camus [6], Geymonat & Suquet [16], Borchers & Sohr [7], and Amrouche & Girault [4, Proposition 2.10]. We shall use here the latter extension, which will be referred to as *J. L. Lions' lemma in $\mathbf{H}^m(\Omega)$* : *Let Ω be a domain in \mathbb{R}^3 and let a distribution $v \in D'(\Omega)$ be such that $\mathbf{grad} v \in \mathbf{H}^m(\Omega)$ for some integer $m \in \mathbb{Z}$. Then $v \in H^{m+1}(\Omega)$.*

Let then $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$ be such that $\nabla_s \mathbf{v} \in \mathbb{H}_s^m(\Omega)$ for some integer $m \in \mathbb{Z}$. The identity

$$(\mathbf{grad}(\partial_k v_i))_j = \partial_j \{(\nabla_s \mathbf{v})_{ik}\} + \partial_k \{(\nabla_s \mathbf{v})_{ij}\} - \partial_i \{(\nabla_s \mathbf{v})_{jk}\}$$

shows that each distribution $\partial_k v_i \in D'(\Omega)$ is such that $\mathbf{grad}(\partial_k v_i) \in \mathbf{H}^{m-1}(\Omega)$. Therefore J. L. Lions' lemma in $\mathbf{H}^{m-1}(\Omega)$ implies that $\partial_k v_i \in H^m(\Omega)$. In other words, each distribution $v_i \in D'(\Omega)$ is such that $\mathbf{grad} v_i \in \mathbf{H}^m(\Omega)$. An application of J. L. Lions' lemma in $\mathbf{H}^m(\Omega)$ then shows that $v_i \in H^{m+1}(\Omega)$, i.e., that $\mathbf{v} \in \mathbf{H}^{m+1}(\Omega)$. \square

The next theorem lists two properties of the operator ∇_s , *considered as acting from the space $\mathbf{L}^2(\Omega)$ into the space $\mathbb{H}_s^{-1}(\Omega)$.*

THEOREM 3.2. *Let Ω be a domain in \mathbb{R}^3 .*

(a) *The operator*

$$\nabla_s : \dot{\mathbf{L}}^2(\Omega) := \mathbf{L}^2(\Omega) / \mathbf{Ker} \nabla_s \rightarrow \mathbb{H}_s^{-1}(\Omega),$$

where for any $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$, $\nabla_s \dot{\mathbf{v}} := \nabla_s \mathbf{w}$ for any $\mathbf{w} \in \dot{\mathbf{v}}$, is an isomorphism from $\dot{\mathbf{L}}^2(\Omega)$ onto $\mathbf{Im} \nabla_s$. Consequently, the space $\mathbf{Im} \nabla_s$ is closed in $\mathbb{H}_s^{-1}(\Omega)$.

(b) *The dual operator of $\nabla_s : \mathbf{L}^2(\Omega) \rightarrow \mathbb{H}_s^{-1}(\Omega)$ is $-\mathbf{div} : \mathbb{H}_{0,s}^1(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ and the dual operator of $\nabla_s : \dot{\mathbf{L}}^2(\Omega) \rightarrow \mathbb{H}_s^{-1}(\Omega)$ is $-\mathbf{div} : \mathbb{H}_{0,s}^1(\Omega) \rightarrow \dot{\mathbf{L}}^2(\Omega)$.*

Proof. (i) It is readily seen that the space

$$\mathbf{K}(\Omega) := \{\mathbf{v} \in \mathbf{H}^{-1}(\Omega); \nabla_s \mathbf{v} \in \mathbb{H}_s^{-1}(\Omega)\},$$

equipped with the norm $\mathbf{v} \rightarrow \|\mathbf{v}\|_{-1,\Omega} + \|\nabla_s \mathbf{v}\|_{-1,\Omega}$ is complete. The identity mapping ι from the space $\mathbf{L}^2(\Omega)$ equipped with $\|\cdot\|_{0,\Omega}$ into the space $\mathbf{K}(\Omega)$ equipped with the above norm is injective, continuous (there clearly exists a constant c such that $\|\mathbf{v}\|_{-1,\Omega} + \|\nabla_s \mathbf{v}\|_{-1,\Omega} \leq c\|\mathbf{v}\|_{0,\Omega}$ for all $\mathbf{v} \in \mathbf{L}^2(\Omega)$), and surjective since the space $\mathbf{K}(\Omega)$ coincides with the space $\mathbf{L}^2(\Omega)$ by the $\mathbb{H}_s^{-1}(\Omega)$ -matrix version of J. L. Lions' lemma established in Theorem 3.1. Therefore the closed graph theorem shows that the inverse mapping ι^{-1} is also continuous, i.e., that *there exists a constant C such that the following Korn inequality in $\mathbf{L}^2(\Omega)$ holds:*

$$\|\mathbf{v}\|_{0,\Omega} \leq C(\|\mathbf{v}\|_{-1,\Omega} + \|\nabla_s \mathbf{v}\|_{-1,\Omega}) \text{ for all } \mathbf{v} \in \mathbf{L}^2(\Omega).$$

(ii) Define the Hilbert space

$$\dot{\mathbf{L}}^2(\Omega) := \mathbf{L}^2(\Omega) / \mathbf{Ker} \nabla_s, \text{ where } \mathbf{Ker} \nabla_s = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \nabla_s \mathbf{v} = \mathbf{0} \text{ in } \mathbb{H}_s^{-1}(\Omega)\}.$$

Recall that we are considering here that *the operator ∇_s maps the space $\mathbf{L}^2(\Omega)$, hence also the quotient space $\dot{\mathbf{L}}^2(\Omega)$, into the space $\mathbb{H}_s^{-1}(\Omega)$* . Note also that, since the space $\mathbf{Ker}\nabla_s$ is finite-dimensional, given any $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$, there exists $\mathbf{w} \in \dot{\mathbf{v}}$ such that $\|\mathbf{w}\|_{0,\Omega} = \|\dot{\mathbf{v}}\|_{0,\Omega}$. We then claim that *there exists a constant \dot{C} such that*

$$\|\dot{\mathbf{v}}\|_{0,\Omega} := \inf_{\mathbf{z} \in \mathbf{Ker}\nabla_s} \|\mathbf{v} + \mathbf{z}\|_{0,\Omega} \leq \dot{C} \|\nabla_s \dot{\mathbf{v}}\|_{-1,\Omega} \text{ for all } \dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega).$$

Assume that such a constant \dot{C} does not exist. Then there exist $\dot{\mathbf{v}}^k \in \dot{\mathbf{L}}^2(\Omega)$ and $\mathbf{w}^k \in \mathbf{L}^2(\Omega)$, $k \geq 0$, such that $\dot{\mathbf{w}}^k = \dot{\mathbf{v}}^k$ and

$$\begin{aligned} \|\mathbf{w}^k\|_{0,\Omega} &= \|\dot{\mathbf{v}}^k\|_{0,\Omega} = 1 \text{ for all } k \geq 0, \\ \|\nabla_s \mathbf{w}^k\|_{-1,\Omega} &= \|\nabla_s \dot{\mathbf{v}}^k\|_{-1,\Omega} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Let $(\mathbf{w}^l)_{l=0}^\infty$ denote a subsequence of the sequence $(\mathbf{w}^k)_{k=0}^\infty$ that converges in $\mathbf{H}^{-1}(\Omega)$ (such a subsequence exists by Rellich' theorem). The Korn inequality established in (i) implies that this subsequence is a Cauchy sequence for the norm $\mathbf{v} \rightarrow \|\mathbf{v}\|_{-1,\Omega} + \|\nabla_s \mathbf{v}\|_{-1,\Omega}$, hence also for the norm $\|\cdot\|_{0,\Omega}$. Consequently, there exists $\mathbf{w} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{w}^l \rightarrow \mathbf{w}$ in $\mathbf{L}^2(\Omega)$ as $l \rightarrow \infty$. Besides, $\nabla_s \mathbf{w} = \mathbf{0}$ in $\mathbf{H}^{-1}(\Omega)$ since $\nabla_s \mathbf{w}^l \rightarrow \nabla_s \mathbf{w}$ in $\mathbf{H}^{-1}(\Omega)$ as $l \rightarrow \infty$. This means that $\mathbf{w} \in \mathbf{Ker}\nabla_s$, so that $\dot{\mathbf{w}}^l \rightarrow \dot{\mathbf{w}} = \dot{\mathbf{0}}$ in $\dot{\mathbf{L}}^2(\Omega)$ as $l \rightarrow \infty$, a contradiction with $\|\dot{\mathbf{w}}^l\|_{0,\Omega} = 1$ for all $l \geq 0$.

(iii) The operator $\nabla_s : \dot{\mathbf{L}}^2(\Omega) \rightarrow \mathbb{H}_s^{-1}(\Omega)$ is injective, continuous, and has an inverse from $\mathbf{Im}\nabla_s \subset \mathbb{H}_s^{-1}(\Omega)$ onto $\dot{\mathbf{L}}^2(\Omega)$ that is also continuous by (ii). In other words, the operator $\nabla_s : \dot{\mathbf{L}}^2(\Omega) \rightarrow \mathbf{Im}\nabla_s$ is an *isomorphism*. Consequently, the space $\mathbf{Im}\nabla_s$ is a Banach space and therefore necessarily a *closed* subspace of $\mathbb{H}_s^{-1}(\Omega)$. This proves (a).

(iv) For any $\mathbf{v} = (v_i) \in \mathbf{L}^2(\Omega)$ and any $\mathbf{e} = (e_{ij}) \in \mathbb{H}_{0,s}^1(\Omega)$,

$$\begin{aligned} \mathbb{H}^{-1}(\Omega) \langle \nabla_s \mathbf{v}, \mathbf{e} \rangle_{\mathbb{H}_{0,s}^1(\Omega)} &=_{H^{-1}(\Omega)} \langle \partial_j v_i, e_{ij} \rangle_{H_0^1(\Omega)} \\ &=_{L^2(\Omega)} \langle v_i, -\partial_j e_{ij} \rangle_{L^2(\Omega)} \\ &=_{\mathbf{L}^2(\Omega)} \langle \mathbf{v}, -\mathbf{div}\mathbf{e} \rangle_{\mathbf{L}^2(\Omega)} \end{aligned}$$

(the first relation uses the symmetry of \mathbf{e}). This proves (b). \square

The two theorems above show that, indeed, *the operator ∇_s may be aptly regarded as the “matrix analog” of the usual gradient operator \mathbf{grad}* . More specifically, the implication established in Theorem 3.1 is the matrix analog of the implication

$$v \in D'(\Omega) \text{ and } \mathbf{grad} v \in \mathbf{H}^m(\Omega) \Rightarrow v \in H^{m+1}(\Omega)$$

used in its proof; the inequalities established in parts (i) and (ii) of the proof of Theorem 3.2 are the matrix analogs of the inequality

$$\|v\|_{0,\Omega} \leq C(\|v\|_{-1,\Omega} + \|\mathbf{grad} v\|_{-1,\Omega}) \text{ for all } v \in L^2(\Omega),$$

established in Nečas [23], and of the inequality

$$\|\dot{v}\|_{L^2(\Omega)/\mathbb{R}} \leq \dot{C} \|\mathbf{grad} v\|_{-1,\Omega} \text{ for all } \dot{v} \in L^2(\Omega)/\mathbb{R},$$

established in Girault & Raviart [17, Corollary 2.1] (as an application of an abstract result due to Peetre [25] and Tartar [26]); finally, part (a) of Theorem 3.2 mimics that \mathbf{grad} is an isomorphism from $L^2(\Omega)/\mathbb{R}$ onto its image in $\mathbf{H}^{-1}(\Omega)$ (cf. Girault & Raviart [17, Corollary 2.4]) and part (b) mimics that the dual operator of $\mathbf{grad}: L^2(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ is $-\text{div}: \mathbf{H}_0^1(\Omega) \rightarrow L^2(\Omega)$.

The next theorem lists two properties of the operator ∇_s , now *considered as acting from $\mathbf{H}_0^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$* .

THEOREM 3.3. *Let Ω be a domain in \mathbb{R}^3 .*

(a) *The operator*

$$\nabla_s : \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{L}_s^2(\Omega)$$

is an isomorphism from $\mathbf{H}_0^1(\Omega)$ onto $\mathbf{Im} \nabla_s$. Consequently, the space $\mathbf{Im} \nabla_s$ is closed in $\mathbb{L}_s^2(\Omega)$.

(b) *The dual operator of $\nabla_s : \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{L}_s^2(\Omega)$ is $-\mathbf{div}: \mathbb{L}_s^2(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$.*

Proof. The proof is similar to that of parts (a) and (b) of Theorem 3.2, and actually simpler since $\mathbf{Ker} \nabla_s = \{\mathbf{0}\}$ in this case. Besides, a well-known elementary computation shows that

$$\left\{ \sum_{i,j} \|\partial_i v_j\|_{0,\Omega}^2 \right\}^{1/2} \leq \sqrt{2} \|\nabla_s \mathbf{v}\|_{0,\Omega} \text{ for all } \mathbf{v} = (v_j) \in \mathbf{H}_0^1(\Omega).$$

Hence the existence of a constant C such that

$$\|\mathbf{v}\|_{1,\Omega} \leq C \|\nabla_s \mathbf{v}\|_{0,\Omega} \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

i.e., the analog to the inequality established in part (ii) of the proof of Theorem 3.2, immediately follows. The rest of the proof is analogous to that of parts (iii) and (iv) of the proof of Theorem 3.2. \square

The operator ∇_s can also be considered as *acting from $\mathbf{H}^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$* , in which case similar arguments show that the operator $\nabla_s : \mathbf{H}^1(\Omega)/\mathbf{Ker} \nabla_s \rightarrow \mathbb{L}_s^2(\Omega)$ is an *isomorphism*, so that $\mathbf{Im} \nabla_s$ is again a *closed* subspace of $\mathbb{L}_s^2(\Omega)$. Interestingly, under the additional assumption that the domain Ω is *simply-connected*, the space $\mathbf{Im} \nabla_s$ can be given an *explicit characterization* in this case, as

$$\mathbf{Im} \nabla_s = \{\mathbf{e} = (e_{ij}) \in \mathbb{L}_s^2(\Omega); \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega)\},$$

thus providing another proof that $\mathbf{Im} \nabla_s$ is closed in $\mathbb{L}_s^2(\Omega)$ when the operator ∇_s is considered as acting from $\mathbf{H}^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$. This characterization of $\mathbf{Im} \nabla_s$, which was first established by Ciarlet & Ciarlet, Jr. [10], is recovered later in this paper as a simple corollary (Theorem 7.2).

4 Extensions of Donati's theorem

As a corollary to Theorem 3.2, we obtain a *first extension of Donati's theorem* (this classical result is recalled in Section 1).

THEOREM 4.1. *Let Ω be a domain in \mathbb{R}^3 and let there be given a matrix field $\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{H}_s^{-1}(\Omega)$ if and only if*

$$\mathbb{H}_s^{-1}(\Omega) \langle \mathbf{e}, \mathbf{s} \rangle_{\mathbb{H}_{0,s}^1(\Omega)} = 0 \text{ for all } \mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega) \text{ satisfying } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \mathbf{L}^2(\Omega).$$

All other vector fields $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ satisfying $\mathbf{e} = \nabla_s \tilde{\mathbf{v}}$ in $\mathbb{H}_s^{-1}(\Omega)$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{a} + \mathbf{b} \wedge \mathbf{id}_\Omega$ for some vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

Proof. Since the dual operator of $\nabla_s : \mathbf{L}^2(\Omega) \rightarrow \mathbb{H}_s^{-1}(\Omega)$ is $-\mathbf{div} : \mathbb{H}_{0,s}^1(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ and the space $\mathbf{Im} \nabla_s$ is closed in $\mathbb{H}_s^{-1}(\Omega)$ (Theorem 3.2), *Banach's closed range theorem* implies that

$$\mathbf{Im} \nabla_s = (\mathbf{Ker}(-\mathbf{div}))^0 = \{\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega); \mathbb{H}_s^{-1}(\Omega) \langle \mathbf{e}, \mathbf{s} \rangle_{\mathbb{H}_{0,s}^1(\Omega)} = 0 \text{ for all } \mathbf{s} \in \mathbf{Ker}(-\mathbf{div})\},$$

which is exactly what the theorem asserts. That all other solutions $\tilde{\mathbf{v}}$ of the equation $\mathbf{e} = \nabla_s \tilde{\mathbf{v}}$ are of the form indicated above follows from the characterization of the space $\mathbf{Ker} \nabla_s$ recalled in Section 3. \square

This extension of Donati's theorem is the “matrix analog” of a well-known characterization of vector fields as gradients of scalar functions (see Girault & Raviart [17, Lemma 2.1]): Let Ω be a domain in \mathbb{R}^3 and let there be given a vector field $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$. Then there exists a function $p \in L^2(\Omega)$ such that $\mathbf{h} = \mathbf{grad} p$ in $\mathbf{H}^{-1}(\Omega)$ if and only if

$$\mathbf{H}^{-1}(\Omega) \langle \mathbf{h}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} = 0 \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \text{ satisfying } \mathbf{div} \mathbf{v} = 0 \text{ in } \mathbf{L}^2(\Omega).$$

In other words, the operator \mathbf{grad} and the spaces $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ appearing in this characterization are replaced in Theorem 4.1 by their “matrix analogs” ∇_s and $\mathbb{H}_s^{-1}(\Omega)$ and $\mathbb{H}_{0,s}^1(\Omega)$ (naturally, the scalar operator \mathbf{div} is replaced by the vector operator \mathbf{div} in the process).

We similarly obtain a *second extension of Donati's theorem*, this time as a corollary to Theorem 3.3.

THEOREM 4.2. *Let Ω be a domain in \mathbb{R}^3 and let there be given a matrix field $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$ if and only if*

$$\int_{\Omega} \mathbf{e} : \mathbf{s} dx = 0 \text{ for all } \mathbf{s} \in \mathbb{L}_s^2(\Omega) \text{ satisfying } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \mathbf{H}^{-1}(\Omega).$$

If this is the case, the vector field \mathbf{v} is unique.

Proof. Since the dual operator of $\nabla_s : \mathbf{H}_0^1(\Omega) \rightarrow \mathbb{L}_s^2(\Omega)$ is $-\mathbf{div} : \mathbb{L}_s^2(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ and the space $\mathbf{Im} \nabla_s$ is closed in $\mathbb{L}_s^2(\Omega)$ (Theorem 3.3), the existence of the vector field \mathbf{v} again follows from Banach's closed range theorem, this time applied to operator ∇_s considered as acting from $\mathbf{H}_0^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$. That $\mathbf{Ker} \nabla_s = \{\mathbf{0}\}$ in this case implies that such a vector field \mathbf{v} is unique. \square

We mention that characterizations similar of Theorems 4.1 and 4.2 have been simultaneously obtained by Geymonat & Krasucki [14], albeit by a different proof for their analog of Theorem 4.2. In addition, they noticed that Theorem 4.1 and its proof can be extended almost *verbatim* to matrix fields $\mathbf{e} \in \mathbb{W}_s^{-1,p}(\Omega)$, $1 < p < \infty$, that satisfy $\mathbb{W}_s^{-1,p}(\Omega) \langle \mathbf{e}, \mathbf{s} \rangle_{\mathbb{W}_{0,s}^{1,q}(\Omega)} = 0$ for all $\mathbf{s} \in \mathbb{W}_{0,s}^{1,q}(\Omega)$ satisfying $\mathbf{div} \mathbf{s} = \mathbf{0}$ in $\mathbf{L}^q(\Omega)$, where $q := \frac{p}{p-1}$ (as expected, the resulting field \mathbf{v} is then found in the space $\mathbf{L}^p(\Omega)$). They also showed how to derive an analog of Theorem 4.2 that can handle the more general “boundary condition” $\mathbf{v} = \mathbf{0}$ on any relatively open subset of the boundary of Ω .

Finally, a *third extension of Donati’s theorem* can also be obtained that “mixes” the regularity assumption of Theorem 4.2 on the matrix field \mathbf{e} with the necessary and sufficient condition of Theorem 4.1. Note that Theorem 4.3 also constitutes an extension of *Ting’s theorem* (recalled in Section 1). The proof given here is considerably simpler, however, than that given by Ting [27] (especially when the domain Ω is not simply-connected).

THEOREM 4.3. *Let Ω be a domain in \mathbb{R}^3 and let there be given a matrix field $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$ if and only if*

$$\int_{\Omega} \mathbf{e} : \mathbf{s} dx = 0 \text{ for all } \mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega) \text{ satisfying } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \mathbf{L}^2(\Omega).$$

All other vector fields $\mathbf{v} \in \mathbf{H}^1(\Omega)$ satisfying $\mathbf{e} = \nabla_s \mathbf{v}$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{a} + \mathbf{b} \wedge \mathbf{id}_{\Omega}$ for some vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

Proof. Let $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ be such that $\int_{\Omega} \mathbf{e} : \mathbf{s} dx = 0$ for all $\mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega)$ satisfying $\mathbf{div} \mathbf{s} = \mathbf{0}$ in $\mathbf{L}^2(\Omega)$. Since $\mathbb{L}_s^2(\Omega) \subset \mathbb{H}_s^{-1}(\Omega)$, Theorem 4.1 shows that there exists $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$, and thus the $\mathbb{L}_s^2(\Omega)$ -matrix version of J. L. Lions’ lemma (Theorem 3.1) further shows that $\mathbf{v} \in \mathbf{H}^1(\Omega)$. The announced relations are therefore sufficient.

If, conversely, $\mathbf{e} = (e_{ij}) = \nabla_s \mathbf{v}$ for some $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, then the symmetry of \mathbf{e} and Green’s formula together imply that

$$\int_{\Omega} e_{ij} s_{ij} dx = \int_{\Omega} (\partial_j v_i) s_{ij} dx = - \int_{\Omega} v_i \partial_j s_{ij} dx \text{ for all } \mathbf{s} = (s_{ij}) \in \mathbb{H}_{0,s}^1(\Omega).$$

Consequently, $\int_{\Omega} e_{ij} s_{ij} dx = 0$ if $\mathbf{div} \mathbf{s} = \mathbf{0}$, and thus the announced relations are also necessary.

The non-uniqueness issue is dealt with as in Theorem 4.1. \square

A comparison between the necessary and sufficient conditions found in Theorems 4.2 and 4.3 shows that the closure with respect to the norm $\|\cdot\|_{0,\Omega}$ of the space $\{\mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega); \mathbf{div} \mathbf{s} = \mathbf{0}\}$ is thus *strictly contained* in the space $\{\mathbf{s} \in \mathbb{L}_s^2(\Omega); \mathbf{div} \mathbf{s} = \mathbf{0}\}$ (otherwise, the vector field \mathbf{v} found in Theorem 4.3 would be necessarily in $\mathbf{H}_0^1(\Omega)$). Naturally, the same conclusion applies *a fortiori* to the closure of the space $\{\mathbf{s} \in \mathbb{D}_s(\Omega); \mathbf{div} \mathbf{s} = \mathbf{0}\}$ appearing in Ting’s theorem.

5 Another approach to linearized elasticity

Let Ω be a domain in \mathbb{R}^3 , now viewed as the reference configuration of a linearly elastic body. This body is characterized by its elasticity tensor field $\mathbf{A} = (A_{ijkl})$ with components $A_{ijkl} \in L^\infty(\Omega)$. It

is assumed as usual that these components satisfy the symmetry relations $A_{ijkl} = A_{jikl} = A_{klij}$, and that there exists a constant $\alpha > 0$ such that $\mathbf{A}(x)\mathbf{t}:\mathbf{t} \geq \alpha\mathbf{t}:\mathbf{t}$ for almost all $x \in \Omega$ and all symmetric matrices $\mathbf{t} = (t_{ij})$ of order three, where $(\mathbf{A}(x)\mathbf{t})_{ij} := A_{ijkl}(x)t_{kl}$. The body is subjected to applied body forces with density $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$. Finally, it is assumed that the linear form $L \in \mathcal{L}(\mathbf{H}^1(\Omega); \mathbb{R})$ defined by $L(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx$ for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ vanishes for all $\mathbf{v} \in \mathbf{Ker} \nabla_s$, where ∇_s is here considered to be acting from $\mathbf{H}^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$, i.e.,

$$\mathbf{Ker} \nabla_s := \{\mathbf{v} \in \mathbf{H}^1(\Omega); \nabla_s \mathbf{v} = \mathbf{0} \text{ in } \mathbb{L}_s^2(\Omega)\}.$$

Then the corresponding *pure traction problem of three-dimensional linearized elasticity* classically consist in finding $\hat{\mathbf{u}} \in \hat{\mathbf{H}}^1(\Omega) := \mathbf{H}^1(\Omega)/\mathbf{Ker} \nabla_s$ such that

$$J(\hat{\mathbf{u}}) = \inf_{\hat{\mathbf{v}} \in \hat{\mathbf{H}}^1(\Omega)} J(\hat{\mathbf{v}}), \text{ where } J(\hat{\mathbf{v}}) := \frac{1}{2} \int_{\Omega} \mathbf{A} \nabla_s \hat{\mathbf{v}} : \nabla_s \hat{\mathbf{v}} dx - L(\hat{\mathbf{v}}).$$

As is well known (see, e.g., Duvaut & Lions [13, Theorem 3.4 in Chapter 3]), this minimization problem has one and only one solution.

Thanks to Theorem 4.3, this problem can be recast as *another quadratic minimization problem*, this time with $\boldsymbol{\varepsilon} := \nabla_s \hat{\mathbf{u}} \in \mathbb{L}_{sym}^2(\Omega)$ as the primary unknown. Note that this minimization problem can be in turn immediately recast as yet another one, this time with the *linearized stress tensor* $\mathbf{A}\boldsymbol{\varepsilon}$ as the primary unknown, since the elasticity tensor field \mathbf{A} is invertible almost everywhere in Ω .

THEOREM 5.1. *Let Ω be a domain in \mathbb{R}^3 . Define the Hilbert space*

$$\mathbf{E}_1(\Omega) := \{\mathbf{e} \in \mathbb{L}_s^2(\Omega); \int_{\Omega} \mathbf{e} : \mathbf{s} dx = 0 \text{ for all } \mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega) \text{ satisfying } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \mathbf{L}^2(\Omega)\},$$

and, for each $\mathbf{e} \in \mathbf{E}_1(\Omega)$, let $\mathcal{F}_1(\mathbf{e})$ denote the unique element in the quotient space $\hat{\mathbf{H}}^1(\Omega)$ that satisfies $\nabla_s \mathcal{F}_1(\mathbf{e}) = \mathbf{e}$ (Theorem 4.3). Then the mapping $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \rightarrow \hat{\mathbf{H}}^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbf{E}_1(\Omega)$ and $\hat{\mathbf{H}}^1(\Omega)$.

The minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbf{E}_1(\Omega)$ such that

$$j_1(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e} \in \mathbf{E}_1(\Omega)} j_1(\mathbf{e}), \text{ where } j_1(\mathbf{e}) := \frac{1}{2} \int_{\Omega} \mathbf{A}\mathbf{e} : \mathbf{e} dx - L \circ \mathcal{F}_1(\mathbf{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \nabla_s \hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is the unique minimizer of the functional J in the space $\hat{\mathbf{H}}^1(\Omega)$.

Proof. By Theorem 4.3, the mapping \mathcal{F}_1 is a bijection between the Hilbert spaces $\mathbf{E}_1(\Omega)$ and $\hat{\mathbf{H}}^1(\Omega)$. Furthermore, its inverse is continuous since there evidently exists a constant c such that

$$\|\nabla_s(\hat{\mathbf{v}})\|_{\mathbb{L}_s^2(\Omega)} = \|\nabla_s(\mathbf{v} + \mathbf{r})\|_{\mathbb{L}_s^2(\Omega)} \leq c\|\mathbf{v} + \mathbf{r}\|_{\mathbf{H}^1(\Omega)}$$

for any $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and any $\mathbf{r} \in \mathbf{Ker} \nabla_s$, so that

$$\|\nabla_s(\hat{\mathbf{v}})\|_{\mathbb{L}_s^2(\Omega)} \leq c \inf_{\mathbf{r} \in \mathbf{Ker} \nabla_s} \|\mathbf{v} + \mathbf{r}\|_{\mathbf{H}^1(\Omega)} = c\|\hat{\mathbf{v}}\|_{\hat{\mathbf{H}}^1(\Omega)}.$$

Hence $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \rightarrow \mathbf{H}^1(\Omega)$ is an isomorphism by the closed graph theorem.

The bilinear form $(\mathbf{e}, \tilde{\mathbf{e}}) \in \mathbf{E}_1(\Omega) \times \mathbf{E}_1(\Omega) \rightarrow \int_{\Omega} \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}} dx \in \mathbb{R}$ and the linear form $\Lambda_1 := L \circ \mathcal{F}_1 : \mathbf{E}_1(\Omega) \rightarrow \mathbb{R}$ thus satisfy all the assumptions of the Lax-Milgram lemma (Λ_1 is continuous since \mathcal{F}_1 is an isomorphism). Consequently, there exists one, and only one, minimizer $\boldsymbol{\varepsilon}$ of the functional j over $\mathbf{E}_1(\Omega)$. That $\dot{\mathbf{u}}$ minimizes the functional J over $\dot{\mathbf{H}}^1(\Omega)$ implies that $\nabla_s \dot{\mathbf{u}}$ minimizes the functional j_1 over $\mathbf{E}_1(\Omega)$. Hence $\boldsymbol{\varepsilon} = \mathbf{e}(\dot{\mathbf{u}})$ since the minimizer is unique. \square

Remarkably, the *Korn inequality in the space $\mathbf{H}^1(\Omega)$* can be recovered as a simple corollary to Theorem 5.1, which thus provides an entirely new proof of this classical inequality (see [13, Theorem 3.1 in Chapter 3]):

THEOREM 5.2. *That the mapping $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \rightarrow \dot{\mathbf{H}}^1(\Omega)$ is an isomorphism (Theorem 5.1) implies Korn's inequality in the space $\mathbf{H}^1(\Omega)$, viz., the existence of a constant C such that*

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C \{ \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\nabla_s \mathbf{v}\|_{\mathbb{L}_s^2(\Omega)}^2 \}^{1/2} \text{ for all } \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Proof. Since $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \rightarrow \dot{\mathbf{H}}^1(\Omega)$ is an isomorphism, there exists a constant \dot{C} such that

$$\|\mathcal{F}_1(\mathbf{e})\|_{\dot{\mathbf{H}}^1(\Omega)} \leq \dot{C} \|\mathbf{e}\|_{\mathbb{L}_s^2(\Omega)} \text{ for all } \mathbf{e} \in \mathbf{E}_1(\Omega),$$

or equivalently, such that

$$\|\dot{\mathbf{v}}\|_{\dot{\mathbf{H}}^1(\Omega)} \leq \dot{C} \|\nabla_s \dot{\mathbf{v}}\|_{\mathbb{L}_s^2(\Omega)} \text{ for all } \dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega).$$

Assume that the announced Korn inequality does not hold. Then there exist $\mathbf{v}^k \in \mathbf{H}^1(\Omega)$, $k \geq 1$, such that

$$\|\mathbf{v}^k\|_{\mathbf{H}^1(\Omega)} = 1 \text{ for all } k \geq 1 \text{ and } (\|\mathbf{v}^k\|_{\mathbf{L}^2(\Omega)} + \|\nabla_s \mathbf{v}^k\|_{\mathbb{L}_s^2(\Omega)}) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let $\mathbf{r}^k \in \mathbf{Ker} \nabla_s$ denote for each $k \geq 1$ the projection of \mathbf{v}^k on $\mathbf{Ker} \nabla_s$ with respect to the inner-product of $\mathbf{H}^1(\Omega)$. This projection therefore satisfies

$$\begin{aligned} \|\mathbf{v}^k - \mathbf{r}^k\|_{\mathbf{H}^1(\Omega)} &= \inf_{\mathbf{r} \in \mathbf{Ker} \nabla_s} \|\mathbf{v}^k - \mathbf{r}\|_{\mathbf{H}^1(\Omega)}, \\ \|\mathbf{v}^k\|_{\mathbf{H}^1(\Omega)}^2 &= \|\mathbf{v}^k - \mathbf{r}^k\|_{\mathbf{H}^1(\Omega)}^2 + \|\mathbf{r}^k\|_{\mathbf{H}^1(\Omega)}^2. \end{aligned}$$

The space $\mathbf{Ker} \nabla_s$ being finite-dimensional, the inequalities $\|\mathbf{r}^k\|_{\mathbf{H}^1(\Omega)} \leq 1$ for all $k \geq 1$ imply the existence of a subsequence $(\mathbf{r}^l)_{l=1}^{\infty}$ that converges in $\mathbf{H}^1(\Omega)$ toward an element $\mathbf{r} \in \mathbf{Ker} \nabla_s$. Besides, the existence of the above constant \dot{C} implies that $\|\mathbf{v}^l - \mathbf{r}^l\|_{\mathbf{H}^1(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$, so that $\|\mathbf{v}^l - \mathbf{r}\|_{\mathbf{H}^1(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$. Hence $\|\mathbf{v}^l - \mathbf{r}\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$, which forces \mathbf{r} to be $\mathbf{0}$, since $\|\mathbf{v}^l\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$ on the other hand. We thus reach the conclusion that $\|\mathbf{v}^l\|_{\mathbf{H}^1(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$, a contradiction. \square

Consider likewise the *pure displacement problem of three-dimensional linearized elasticity*, which classically consists in finding $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$J(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} J(\mathbf{v}), \text{ where } J(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \mathbf{A} \nabla_s \mathbf{v} : \nabla_s \mathbf{v} dx - L(\mathbf{v}),$$

where again $L(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx$ for some $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ (no extra condition need to be imposed on the linear form L in this case, since $\mathbf{Ker} \nabla_s = \{\mathbf{0}\}$ in $\mathbf{H}_0^1(\Omega)$).

Thanks to Theorem 4.2, this problem can be again recast as *another quadratic minimization problem*, this time with $\boldsymbol{\varepsilon} := \nabla_s \mathbf{u} \in \mathbb{L}_s^2(\Omega)$ as the primary unknown:

THEOREM 5.3. *Let Ω be a domain in \mathbb{R}^3 . Define the Hilbert space*

$$\mathbf{E}_2(\Omega) := \{\mathbf{e} \in \mathbb{L}_s^2(\Omega); \int_{\Omega} \mathbf{e} : \mathbf{s} dx = 0 \text{ for all } \mathbf{s} \in \mathbb{L}_s^2(\Omega) \text{ satisfying } \mathbf{div} \mathbf{s} = \mathbf{0} \text{ in } \mathbf{H}^{-1}(\Omega)\},$$

and, for each $\mathbf{e} \in \mathbf{E}_2(\Omega)$, let $\mathfrak{F}_2(\mathbf{e})$ denote the unique element in the space $\mathbf{H}_0^1(\Omega)$ that satisfies $\nabla_s \mathfrak{F}_2(\mathbf{e}) = \mathbf{e}$ (Theorem 4.2). Then the mapping $\mathfrak{F}_2 : \mathbf{E}_2(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbf{E}_2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$.

The minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbf{E}_2(\Omega)$ such that

$$j_2(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e} \in \mathbf{E}_2(\Omega)} j_2(\mathbf{e}), \text{ where } j_2(\mathbf{e}) := \frac{1}{2} \int_{\Omega} \mathbf{A} \mathbf{e} : \mathbf{e} dx - L \circ \mathfrak{F}_2(\mathbf{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \nabla_s \mathbf{u}$, where \mathbf{u} is the unique minimizer of the functional J in the space $\mathbf{H}_0^1(\Omega)$.

Proof. The proof is similar to that of Theorem 5.1 and, for this reason, is omitted. \square

Naturally, the classical *Korn inequality in the space $\mathbf{H}_0^1(\Omega)$* , viz., the existence of a constant C_0 such that

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C_0 \{ \|\mathbf{v}\|_{\mathbb{L}^2(\Omega)}^2 + \|\nabla_s \mathbf{v}\|_{\mathbb{L}_s^2(\Omega)}^2 \}^{1/2} \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

could be also recovered in a manner similar to that used in Theorem 5.2. This observation does not have much significance, however, since (as already noted in the proof of Theorem 3.3) it is well known that elementary computations directly show that this inequality holds with $C_0 = \sqrt{2}$.

6 The operator CURLCURL

Let Ω be an open subset of \mathbb{R}^3 . For any matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$, the matrix field **CURLCURL** $\mathbf{e} \in \mathbb{D}'(\Omega)$ is defined by

$$\mathbf{CURLCURL} \mathbf{e} := \mathbf{CURL} (\mathbf{CURL} \mathbf{e}),$$

or equivalently by

$$(\mathbf{CURLCURL} \mathbf{e})_{ij} := \varepsilon_{ikl} \varepsilon_{jmn} \partial_{ln} e_{km}.$$

Another objective of this paper is to show that the operator **CURLCURL** : $\mathbb{D}'_s(\Omega) \rightarrow \mathbb{D}'(\Omega)$ defined in this fashion is in various ways the “matrix analog” of the “vector” operator **curl** : $\mathbf{D}'(\Omega) \rightarrow \mathbf{D}'(\Omega)$. The next theorem, which lists some algebraic properties of this operator, includes some identities that constitute a first contribution to this objective.

THEOREM 6.1. *Let Ω be any open subset of \mathbb{R}^3 . The operator **CURLCURL** possesses the following properties:*

(a) *For any matrix field $\mathbf{e} \in \mathbb{D}'_s(\Omega)$,*

$$\begin{aligned}\mathbf{CURLCURL} \mathbf{e} &= (\mathbf{CURLCURL} \mathbf{e})^T \text{ in } \mathbb{D}'(\Omega), \\ \mathbf{div}(\mathbf{CURLCURL} \mathbf{e}) &= \mathbf{0} \text{ in } \mathbf{D}'(\Omega), \\ \text{tr}(\mathbf{CURLCURL} \mathbf{e}) &= \Delta(\text{tr} \mathbf{e}) - \mathbf{div}(\mathbf{div} \mathbf{e}) \text{ in } \mathbf{D}'(\Omega).\end{aligned}$$

(b) *Given any matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'_s(\Omega)$, let*

$$R_{ijkl}(\mathbf{e}) := \partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} \text{ in } \mathbf{D}'(\Omega).$$

Then each distribution $R_{ijkl}(\mathbf{e})$ that does not identically vanish is equal to some distribution $(\mathbf{CURLCURL} \mathbf{e})_{pq}$ for appropriate indices p and q , and conversely. Consequently, the eighty-one relations

$$R_{ijkl}(\mathbf{e}) = 0 \text{ in } \mathbf{D}'(\Omega)$$

are equivalent to the six relations $(\mathbf{CURLCURL} \mathbf{e})_{mn} = 0$ in $\mathbf{D}'(\Omega)$, $m \leq n$, i.e., to

$$\mathbf{CURLCURL} \mathbf{e} = \mathbf{0} \text{ in } \mathbb{D}'_s(\Omega).$$

(c) *For any vector field $\mathbf{v} \in \mathbf{D}'(\Omega)$,*

$$\mathbf{CURLCURL} (\nabla_s \mathbf{v}) = \mathbf{0} \text{ in } \mathbb{D}'(\Omega).$$

Proof. First, let a matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$ be given. Then we immediately obtain

$$(\mathbf{CURLCURL} \mathbf{e})_{ji} = \varepsilon_{jmn} \varepsilon_{ikl} \partial_{nl} e_{mk} = (\mathbf{CURLCURL} (\mathbf{e}^T))_{ij}.$$

Let a matrix field $\mathbf{e} \in \mathbb{D}'_s(\Omega)$ be given. Noting that the j -th component of the vector $\mathbf{div}(\mathbf{CURLCURL} \mathbf{e})$ is the divergence of the j -th column vector of $(\mathbf{CURLCURL} \mathbf{e})^T = \mathbf{CURLCURL} \mathbf{e}$, we next infer that

$$(\mathbf{div}(\mathbf{CURLCURL} \mathbf{e}))_j = \text{div}(\mathbf{curl} \mathbf{v}_j) = 0,$$

where \mathbf{v}_j denotes the j -th column vector of $(\mathbf{CURL} \mathbf{e})^T$.

Noting that

$$\varepsilon_{ikl} \varepsilon_{imn} = \delta_{km} \delta_{ln} - \delta_{kn} \delta_{lm},$$

we finally obtain

$$\begin{aligned}\text{tr}(\mathbf{CURLCURL} \mathbf{e}) &= \varepsilon_{ikl} \varepsilon_{imn} \partial_{ln} e_{km} = (\delta_{km} \delta_{ln} - \delta_{kn} \delta_{lm}) \partial_{ln} e_{km} \\ &= \partial_{ll} e_{kk} - \partial_n (\partial_m e_{nm}) = \Delta(\text{tr} \mathbf{e}) - \mathbf{div}(\mathbf{div} \mathbf{e}),\end{aligned}$$

and thus all the identities announced in (a) are established.

Second, let again a matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'_s(\Omega)$ be given and let $\mathbf{q} = (q_{ij}) := \mathbf{CURLCURL} \mathbf{e}$. Then a direct computation shows that

$$\begin{aligned} q_{11} &= R_{2323}(\mathbf{e}), \quad q_{12} = R_{2331}(\mathbf{e}), \quad q_{13} = R_{1223}(\mathbf{e}), \\ q_{22} &= R_{1313}(\mathbf{e}), \quad q_{23} = R_{1312}(\mathbf{e}), \quad q_{33} = R_{1212}(\mathbf{e}). \end{aligned}$$

Taking also into account the relations

$$\begin{aligned} R_{ijkl}(\mathbf{e}) &= 0 \text{ if } i = j \text{ or } k = l, \\ R_{ijkl}(\mathbf{e}) &= R_{klij}(\mathbf{e}) = -R_{jikl}(\mathbf{e}) = -R_{ijlk}(\mathbf{e}), \end{aligned}$$

we thus easily conclude that all the distributions $R_{ijkl}(\mathbf{e})$ that do not identically vanish are known if and only if the six ones appearing above (i.e., $R_{2323}(\mathbf{e}), \dots, R_{1212}(\mathbf{e})$) are known. This proves (b).

Third, let a vector field $\mathbf{v} = (v_j) \in \mathbf{D}'(\Omega)$ be given. As shown above, $\mathbf{CURLCURL}(\mathbf{e}^T) = (\mathbf{CURLCURL} \mathbf{e})^T$ for any $\mathbf{e} \in \mathbb{D}(\Omega)$; consequently,

$$\begin{aligned} \mathbf{CURLCURL}(\nabla_s \mathbf{v}) &= \frac{1}{2} \mathbf{CURLCURL}(\nabla \mathbf{v}^T + \nabla \mathbf{v}) \\ &= \frac{1}{2} \mathbf{CURLCURL}(\nabla \mathbf{v}^T) + \frac{1}{2} (\mathbf{CURLCURL}(\nabla \mathbf{v}^T))^T. \end{aligned}$$

Since the j -th column vector of $\nabla \mathbf{v}^T$ is $\mathbf{grad} v_j$, the j -th column vector of $\mathbf{CURL}(\nabla \mathbf{v}^T)$ is $\mathbf{curl} \mathbf{grad} v_j = \mathbf{0}$. Hence (c) is proved. \square

Note that the relations

$$\mathbf{div}(\mathbf{CURLCURL} \mathbf{e}) = \mathbf{0} \text{ and } \mathbf{CURLCURL}(\nabla_s \mathbf{v}) = \mathbf{0},$$

established in Theorem 6.1 for arbitrary matrix fields $\mathbf{e} \in \mathbb{D}'_s(\Omega)$ and vector fields $\mathbf{v} \in \mathbf{D}'(\Omega)$, are indeed the “matrix analogs” of the well-known relations

$$\mathbf{div}(\mathbf{curl} \mathbf{v}) = 0 \text{ and } \mathbf{curl}(\mathbf{grad} v) = \mathbf{0},$$

which hold for arbitrary vector fields $\mathbf{v} \in \mathbf{D}'(\Omega)$ and distributions $v \in D'(\Omega)$.

7 An extension of Saint Venant’s theorem and its relation to Poincaré’s lemma

Let Ω be any open subset in \mathbb{R}^3 . Given any vector field $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, Theorem 6.1 shows that $\mathbf{CURLCURL}(\nabla_s \mathbf{v}) = \mathbf{0}$ in $\mathbb{D}'_s(\Omega)$, or equivalently, that the *Saint Venant compatibility conditions* $R_{ijkl}(\nabla_s \mathbf{v}) = 0$ hold in $D'(\Omega)$.

As recalled in Section 1, it has been known for a long time that the following converse, known as *Saint Venant’s theorem*, holds for smooth enough matrix fields: *Let Ω be a simply-connected open subset of \mathbb{R}^3 . Assume that, for some integer $m \geq 2$, a matrix field $\mathbf{e} \in \mathbb{C}_s^m(\Omega)$ satisfies the relations $R_{ijkl}(\mathbf{e}) = 0$ in Ω . Then there exists a vector field $\mathbf{v} \in \mathbf{C}^{m+1}(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in Ω .*

We now show that the same Saint Venant compatibility conditions $R_{ijkl}(\mathbf{e}) = 0$ remain sufficient in a much weaker sense, according to the following *Saint Venant’s theorem in $\mathbb{H}_s^{-1}(\Omega)$* .

THEOREM 7.1. *Let Ω be a simply-connected domain in \mathbb{R}^3 and let $\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega)$ be a matrix field that satisfies*

$$\mathbf{CURLCURL} \mathbf{e} = \mathbf{0} \text{ in } \mathbb{H}_s^{-3}(\Omega).$$

Then there exists a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ that satisfies

$$\mathbf{e} = \nabla_s \mathbf{v} \text{ in } \mathbb{H}_s^{-1}(\Omega).$$

All other vector fields $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ satisfying $\mathbf{e} = \nabla_s \tilde{\mathbf{v}}$ in $\mathbb{H}_s^{-1}(\Omega)$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{a} + \mathbf{b} \wedge \mathbf{id}_\Omega$ for some vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

Proof. (i) We know from Theorem 3.2 that $-\mathbf{div}: \mathbb{H}_{0,s}^1(\Omega) \rightarrow \dot{\mathbf{L}}^2(\Omega) = \mathbf{L}^2(\Omega)/\mathbf{Ker} \nabla_s$ is the dual operator of $\nabla_s: \dot{\mathbf{L}}^2(\Omega) \rightarrow \mathbb{H}_s^{-1}(\Omega)$ and that the operator $\nabla_s: \dot{\mathbf{L}}^2(\Omega) \rightarrow \mathbf{Im} \nabla_s = \mathbb{V}^0$, where $\mathbb{V} := \mathbf{Ker}(-\mathbf{div}) \subset \mathbb{H}_{0,s}^1(\Omega)$, is an isomorphism. Consequently, the operator $-\mathbf{div}: (\mathbb{V}^0)' \rightarrow \dot{\mathbf{L}}^2(\Omega)$ is also an *isomorphism*. Besides, the inclusion $\mathbb{V}^0 \subset \mathbb{H}_s^{-1}(\Omega) = (\mathbb{H}_{0,s}^1(\Omega))'$ implies that $(\mathbb{V}^0)'$ can be identified with a (closed) subspace of $\mathbb{H}_{0,s}^1(\Omega)$ (as a Hilbert space, $\mathbb{H}_{0,s}^1(\Omega)$ is uniformly convex, so that the extension from $(\mathbb{V}^0)'$ to $\mathbb{H}_{0,s}^1(\Omega)$ provided by the Hahn-Banach theorem can be defined in a unique fashion).

We thus reach two conclusions. First, *given any element $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$, there exists a unique matrix field $\mathbf{q}(\dot{\mathbf{v}}) \in (\mathbb{V}^0)' \subset \mathbb{H}_{0,s}^1(\Omega)$ such that*

$$-\mathbf{div} \mathbf{q}(\dot{\mathbf{v}}) = \dot{\mathbf{v}} \text{ in } \dot{\mathbf{L}}^2(\Omega).$$

Second, *there exists a constant $\beta > 0$ such that*

$$\beta \|\mathbf{q}(\dot{\mathbf{v}})\|_{1,\Omega} \leq \|\dot{\mathbf{v}}\|_{0,\Omega} \text{ for all } \dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega).$$

(ii) Define two bilinear forms $a: \mathbb{H}_{0,s}^1(\Omega) \times \mathbb{H}_{0,s}^1(\Omega) \rightarrow \mathbb{R}$ and $b: \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}_{0,s}^1(\Omega) \rightarrow \mathbb{R}$ by

$$a(\mathbf{q}, \mathbf{r}) := \int_{\Omega} \partial_k q_{ij} \partial_k r_{ij} \, dx \text{ for all } (\mathbf{q}, \mathbf{r}) = ((q_{ij}), (r_{ij})) \in \mathbb{H}_{0,s}^1(\Omega) \times \mathbb{H}_{0,s}^1(\Omega),$$

$$b(\dot{\mathbf{v}}, \mathbf{q}) := - \int_{\Omega} v_i \partial_j q_{ij} \, dx \text{ for all } (\dot{\mathbf{v}}, \mathbf{q}) = ((v_i), (q_{ij})) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}_{0,s}^1(\Omega).$$

The bilinear form b is indeed unambiguously defined, because the symmetry of \mathbf{q} implies that

$$- \int_{\Omega} v_i \partial_j q_{ij} \, dx =_{\mathbb{H}_s^{-1}(\Omega)} \langle \nabla_s \mathbf{v}, \mathbf{q} \rangle_{\mathbb{H}_{0,s}^1(\Omega)} \text{ for all } (\mathbf{v}, \mathbf{q}) = ((v_i), (q_{ij})) \in \mathbf{L}^2(\Omega) \times \mathbb{H}_{0,s}^1(\Omega);$$

consequently, $- \int_{\Omega} v_i \partial_j q_{ij} \, dx = 0$ if $\mathbf{v} = (v_i) \in \mathbf{Ker} \nabla_s$. Clearly, the two bilinear forms are continuous and the bilinear form a is $\mathbb{H}_{0,s}^1(\Omega)$ -elliptic. Furthermore, the bilinear form b satisfies the *Babuška-Brezzi inf-sup condition*:

$$\inf_{\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)} \frac{1}{\|\dot{\mathbf{v}}\|_{0,\Omega}} \left(\sup_{\mathbf{q} \in \mathbb{H}_{0,s}^1(\Omega)} \frac{b(\dot{\mathbf{v}}, \mathbf{q})}{\|\mathbf{q}\|_{1,\Omega}} \right) \geq \beta,$$

where $\beta > 0$ is the constant found in (i). To see this, we simply note that for any $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$,

$$b(\dot{\mathbf{v}}, \dot{\mathbf{q}}(\dot{\mathbf{v}})) = - \int_{\Omega} \dot{v}_i \partial_j q_{ij}(\dot{\mathbf{v}}) \, dx = \|\dot{\mathbf{v}}\|_{0,\Omega}^2,$$

where $\mathbf{q}(\dot{\mathbf{v}}) \in \mathbb{H}_{0,s}^1(\Omega)$ is defined as in (i), so that

$$\sup_{\mathbf{q} \in \mathbb{H}_{0,s}^1(\Omega)} \frac{b(\dot{\mathbf{v}}, \mathbf{q})}{\|\mathbf{q}\|_{1,\Omega}} \geq \frac{b(\dot{\mathbf{v}}, \dot{\mathbf{q}}(\dot{\mathbf{v}}))}{\|\mathbf{q}(\dot{\mathbf{v}})\|_{0,\Omega}} = \frac{\|\dot{\mathbf{v}}\|_{0,\Omega}^2}{\|\mathbf{q}(\dot{\mathbf{v}})\|_{0,\Omega}} \geq \beta \|\dot{\mathbf{v}}\|_{0,\Omega}.$$

All the assumptions of the *Babuška-Brezzi theorem* (see Babuška [5] and Brezzi [8]) are thus satisfied. Consequently, *given any element $\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega)$, there exists a unique solution $(\dot{\mathbf{u}}, \mathbf{q}) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}_{0,s}^1(\Omega)$ to the equations*

$$\begin{aligned} a(\mathbf{q}, \mathbf{r}) + b(\dot{\mathbf{u}}, \mathbf{r}) &=_{\mathbb{H}_s^{-1}(\Omega)} \langle \mathbf{e}, \mathbf{r} \rangle_{\mathbb{H}_{0,s}^1(\Omega)} \text{ for all } \mathbf{r} \in \mathbb{H}_{0,s}^1(\Omega), \\ b(\dot{\mathbf{v}}, \mathbf{q}) &= 0 \text{ for all } \dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega), \end{aligned}$$

or equivalently, to the equations

$$\begin{aligned} -\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} &= \mathbf{e} \text{ in } \mathbb{H}_s^{-1}(\Omega), \\ \operatorname{div} \mathbf{q} &= \mathbf{0} \text{ in } \dot{\mathbf{L}}^2(\Omega). \end{aligned}$$

(iii) Assume that the element $\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega)$ appearing in the right-hand side of the penultimate equation satisfies *in addition* $\mathbf{CURLCURL} \mathbf{e} = \mathbf{0}$ in $\mathbb{H}_s^{-3}(\Omega)$, so that, by Theorem 6.1 (c),

$$\Delta(\mathbf{CURLCURL} \mathbf{q}) = \mathbf{CURLCURL}(\Delta \mathbf{q}) = \mathbf{CURLCURL}(\nabla_s \dot{\mathbf{u}} - \mathbf{e}) = \mathbf{0} \text{ in } \mathbb{H}_s^{-3}(\Omega).$$

The *hypoellipticity of the Laplacian* (see, e.g., Dautray & Lions [12, Section 2 in Chapter 5]) then implies that $\mathbf{CURLCURL} \mathbf{q} \in \mathbb{C}_s^\infty(\Omega)$, and Theorem 6.1 (a) in turn shows that

$$\Delta(\operatorname{tr} \mathbf{q}) = \operatorname{div}(\operatorname{div} \mathbf{q}) + \operatorname{tr}(\mathbf{CURLCURL} \mathbf{q}) = \operatorname{tr}(\mathbf{CURLCURL} \mathbf{q}) \in \mathbb{C}^\infty(\Omega).$$

Hence $\operatorname{tr} \mathbf{q} \in C^\infty(\Omega)$, again by the hypoellipticity of the Laplacian.

By Theorem 6.1 (b), any distribution $R_{ijkl}(\mathbf{q})$ that does not identically vanish is equal to some distribution $(\mathbf{CURLCURL} \mathbf{q})_{mn}$ for *ad hoc* indices m and n . In particular then, for all indices i and k ,

$$\begin{aligned} R_{ikl}(\mathbf{q}) &= \partial_l q_{ik} + \partial_{ki} q_{ll} - \partial_i(\partial_l q_{kl}) - \partial_k(\partial_l q_{il}) \\ &= \{\Delta q_{ik} + \partial_{ik}(\operatorname{tr} \mathbf{q})\} \in C^\infty(\Omega), \end{aligned}$$

which implies that $\Delta \mathbf{q} \in \mathbb{C}_s^\infty(\Omega)$.

To sum up, we have shown that, *if $\mathbf{CURLCURL} \mathbf{e} = \mathbf{0}$ in $\mathbb{H}_s^{-3}(\Omega)$, the second argument \mathbf{q} of the solution $(\dot{\mathbf{u}}, \mathbf{q}) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}_{0,s}^1(\Omega)$ to the equations $-\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} = \mathbf{e}$ in $\mathbb{H}_s^{-1}(\Omega)$ and $\operatorname{div} \mathbf{q} = \mathbf{0}$ in $\dot{\mathbf{L}}^2(\Omega)$ satisfies*

$$\Delta \mathbf{q} \in \mathbb{C}_s^\infty(\Omega) \text{ and } \mathbf{CURLCURL}(\Delta \mathbf{q}) = \mathbf{0} \text{ in } \Omega.$$

(iv) Since the matrix field $\Delta \mathbf{q} \in \mathbb{C}_s^\infty(\Omega)$ satisfies $\mathbf{CURLCURL}(\Delta \mathbf{q}) = \mathbf{0}$ in the simply-connected open set Ω , the *classical Saint Venant theorem* (i.e., that holds for smooth functions; see the beginning of this section) shows that there exists a vector field $\mathbf{w} \in \mathbb{C}^\infty(\Omega)$ such that $\Delta \mathbf{q} = \nabla_s \mathbf{w}$ in Ω (this is the only place where the simple-connectedness of Ω is used).

The vector field $\mathbf{w} \in \mathbf{C}^\infty(\Omega) \subset \mathbf{D}'(\Omega)$ therefore satisfies $\nabla_s \mathbf{w} = \{\nabla_s \dot{\mathbf{u}} - \mathbf{e}\} \in \mathbb{H}_s^{-1}(\Omega)$. Consequently, the $\mathbb{H}_s^{-1}(\Omega)$ -matrix version of *J. L. Lions' lemma* (Theorem 3.1) shows that $\mathbf{w} \in \mathbf{L}^2(\Omega)$. We have thus established that $\dot{\mathbf{v}} := \{\dot{\mathbf{u}} - \dot{\mathbf{w}}\} \in \dot{\mathbf{L}}^2(\Omega)$ satisfies $\mathbf{e} = \nabla_s \dot{\mathbf{v}}$ in $\mathbb{H}_s^{-1}(\Omega)$, which concludes the existence proof.

The non-uniqueness issue is dealt with as in Theorem 4.1. □

It is worth noticing that *the equations* (encountered in part (ii) of the above proof)

$$\begin{aligned} -\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} &= \mathbf{e} \text{ in } \mathbb{H}_s^{-1}(\Omega), \\ \operatorname{div} \mathbf{q} &= \mathbf{0} \text{ in } \dot{\mathbf{L}}^2(\Omega), \end{aligned}$$

constitute the “matrix analog” of the familiar stationary Stokes problem. We recall that this problem consists in finding a pair $(\dot{p}, \mathbf{u}) \in \dot{L}^2(\Omega) \times \mathbf{H}_0^1(\Omega)$, where $\dot{L}^2(\Omega) := L^2(\Omega)/\mathbb{R}$, that satisfies the equations

$$\begin{aligned} -\nu \Delta \mathbf{u} + \operatorname{grad} \dot{p} &= \mathbf{f} \text{ in } \mathbf{H}^{-1}(\Omega), \\ \operatorname{div} \mathbf{u} &= \mathbf{0} \text{ in } \dot{L}^2(\Omega). \end{aligned}$$

Here, p is the unknown pressure inside an incompressible viscous fluid of viscosity ν and density equal to one, $\mathbf{u} = (u_i)$ is the unknown velocity field of the fluid, and the given vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ accounts for the body forces applied to the fluid. This observation explains in particular why the existence theory used in part (ii) resembles that used for the Stokes problem (see Girault & Raviart [17, Section 5.1])

In the same vein, we emphasize that the *Saint Venant theorem* in $\mathbb{H}_s^{-1}(\Omega)$ (Theorem 7.1) constitutes the matrix analog of the *Poincaré lemma* in $\mathbf{H}^{-1}(\Omega)$, which takes the following form: *Let Ω be a simply-connected domain in \mathbb{R}^3 . If a vector field $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ satisfies $\operatorname{curl} \mathbf{h} = \mathbf{0}$ in $\mathbf{H}^{-2}(\Omega)$, then there exists a function $p \in L^2(\Omega)$ such that $\mathbf{h} = \operatorname{grad} p$* (Poincaré’s lemma in $\mathbf{H}^{-1}(\Omega)$, which is due to Ciarlet & Ciarlet, Jr. [10], was later given a different and simpler proof by Kesavan [19]). In other words, the “vector” operators curl and grad appearing in Poincaré’s lemma are “replaced” in Theorem 7.1 by their “matrix analogs” $\operatorname{CURLCURL}$ and ∇_s .

As expected, a *Saint Venant’s theorem* in $\mathbb{L}_s^2(\Omega)$, i.e., similar to that of Theorem 7.1 but with a “shift by +1” in the regularities of both fields \mathbf{e} and \mathbf{v} , likewise holds as a corollary to Theorem 7.1:

THEOREM 7.2. *Let Ω be a simply-connected domain in \mathbb{R}^3 and let $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ be a matrix field that satisfies*

$$\operatorname{CURLCURL} \mathbf{e} = \mathbf{0} \text{ in } \mathbb{H}_s^{-2}(\Omega).$$

Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ that satisfies

$$\mathbf{e} = \nabla_s \mathbf{v} \text{ in } \mathbb{L}_s^2(\Omega).$$

Proof. Since $\mathbb{L}_s^2(\Omega) \subset \mathbb{H}_s^{-1}(\Omega)$, Theorem 7.1 shows that there exists $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$. Theorem 3.1 then implies that $\mathbf{v} \in \mathbf{H}^1(\Omega)$. □

Saint Venant’s theorem in $\mathbb{L}_s^2(\Omega)$ is due to Ciarlet & Ciarlet, Jr. [10]. More recently, another proof of this result was given by Geymonat & Krasucki [14], by means of arguments that may

not be able to provide a proof of Saint Venant's theorem in $\mathbb{H}_s^{-1}(\Omega)$ (Theorem 7.1), however. See also Geymonat & Krasucki [15], who showed how Saint Venant's theorem in $\mathbb{L}_s^2(\Omega)$ can be extended to non simply-connected domains Ω by means of *Beltrami's functions*.

In Ciarlet & Ciarlet, Jr. [10], it is also shown how Saint Venant's theorem in $\mathbb{L}_s^2(\Omega)$ can be put to use so as to provide yet *another reformulation of the pure traction problem of linearized three-dimensional elasticity posed over simply-connected domains*, which thus constitutes an alternative to that found in Theorem 5.1.

To conclude our analysis, we return to Saint Venant theorem in $\mathbb{H}_s^{-1}(\Omega)$ (Theorem 7.1) and put it in another perspective. To this end, we first record the following equivalence, which is due to Kesavan [19]: *Let Ω be a simply-connected domain in \mathbb{R}^3 . Then the following statements are equivalent:*

- (a) *If $v \in D'(\Omega)$ is such that $\mathbf{grad} v \in \mathbf{H}^{-1}(\Omega)$, then $v \in L^2(\Omega)$*
- (b) *If $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ satisfies $\mathbf{curl} \mathbf{h} = \mathbf{0}$ in $\mathbf{H}^{-2}(\Omega)$, then $\mathbf{h} = \mathbf{grad} p$ for some $p \in L^2(\Omega)$.*

In other words, *J. L. Lions' lemma in $\mathbf{H}^{-1}(\Omega)$ (statement (a)) is equivalent to Poincaré's lemma in $\mathbf{H}^{-1}(\Omega)$ (statement (b)).*

We now show that, likewise, the $\mathbb{H}_s^{-1}(\Omega)$ -matrix version of *J. L. Lions' lemma* (established in Theorem 3.1; statement (a) in the next theorem) *is equivalent to Saint Venant's theorem in $\mathbb{H}_s^{-1}(\Omega)$ (established in Theorem 7.1; statement (b) in the next theorem):*

THEOREM 7.3. *Let Ω be a simply-connected domain in \mathbb{R}^3 . The following statements are equivalent:*

- (a) *If $\mathbf{w} \in \mathbf{D}'(\Omega)$ satisfies $\nabla_s \mathbf{w} \in \mathbb{H}_s^{-1}(\Omega)$, then $\mathbf{w} \in \mathbf{L}^2(\Omega)$.*
- (b) *If $\mathbf{e} \in \mathbb{H}_s^{-1}(\Omega)$ satisfies $\mathbf{CURLCURL} \mathbf{e} = \mathbf{0}$ in $\mathbb{H}_s^{-3}(\Omega)$, then $\mathbf{e} = \nabla_s \mathbf{v}$ for some $\mathbf{v} \in \mathbf{L}^2(\Omega)$.*

Proof. Theorem 3.1 is used in part (iv) of the proof of Theorem 7.1. Hence (a) implies (b).

Assume next that (b) holds and let $\mathbf{w} \in \mathbf{D}'(\Omega)$ be such that $\nabla_s \mathbf{w} \in \mathbb{H}_s^{-1}(\Omega)$. Noting that $\mathbf{CURLCURL}(\nabla_s \mathbf{w}) = \mathbf{0}$ by Theorem 6.1 (c), we infer from (b) that $\nabla_s \mathbf{w} = \nabla_s \mathbf{v}$ for some $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Hence $(\mathbf{w} - \mathbf{v}) \in \mathbf{Ker} \nabla_s \subset \mathbf{L}^2(\Omega)$ and thus $\mathbf{w} \in \mathbf{L}^2(\Omega)$. Hence (b) implies (a). \square

Theorem 7.3 constitutes another evidence that *Saint Venant theorem in $\mathbb{H}_s^{-1}(\Omega)$ is indeed the matrix analog of Poincaré's lemma in $\mathbf{H}^{-1}(\Omega)$.*

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