Mathematical Models and Methods in Applied Sciences

ON THE GENERALIZED VON KÁRMÁN EQUATIONS AND THEIR APPROXIMATION

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We consider here the "generalized von Kármán equations", which constitute a mathematical model for a nonlinearly elastic plate subjected to boundary conditions "of von Kármán type" only on a portion of its lateral face, the remaining portion being free. As already shown elsewhere, solving these equations amounts to solving a "cubic" operator equation, which generalizes an equation introduced by M. S. Berger and P. Fife. Two noticeable features of this equation, which are not encountered in the "classical" von Kármán equations are the lack of strict positivity of its cubic part and the lack of an associated functional. We establish here the convergence of a conforming finite element approximation to these equations. The proof relies in particular on a compactness method due to J.L. Lions and on Brouwer's fixed point theorem. This convergence proof provides in addition an existence proof for the original problem.

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1. Introduction

Using the method of formal asymptotic expansions with the thickness as the "small" parameter, Ciarlet[5] has shown that the classical two-dimensional *von Kármán equations* correspond to a specific class of three-dimensional *boundary conditions*, henceforth called "*of von Kármán type*": The applied surface forces along the lateral face of the plate should be such that only their resultant after integration across the thickness is known and this resultant should be in the plane containing the middle surface of the plate. As a consequence, the transverse segments along the lateral face can undergo only translations, *a priori* of unknown magnitude, in the plane spanned by the middle surface.

In *ibid.*, it was shown in particular that the classical von Kármán equations are obtained only if such boundary conditions of von Kármán type hold over the *entire* lateral face of the plate.

More recently, Ciarlet and Gratie[7] considered nonlinearly elastic plates where such boundary conditions of von Kármán type hold only on a *portion* of the lateral face, the remaining portion being subjected to boundary conditions of free edge. They then showed that a formal asymptotic analysis of the three-dimensional equations leads in this case to "*generalized von Kármán equations*", i.e., a two-dimensional boundary value problem that contains the classical von Kármán equations as a special case. That this generalization is possible hinges on the somewhat unexpected result that the boundary conditions for the *Airy function* can still be determined on the entire boundary solely from the given data.

The generalized von Kármán equations found in this fashion can then be studied on their own from the mathematical and numerical viewpoints, i.e., as regards the existence, uniqueness or non uniqueness, of their solutions and the conception of efficient numerical schemes for their approximation. In this fashion, physical situations of outstanding practical interest, such as the buckling of a rectangular plate subjected to compressive forces on opposite edges, is put on a sound basis, as regards their mathematical modeling, the study of the buckling phenomena, their numerical simulation, etc.

The main objective of the present paper, whose content is described below, is to establish the *convergence of a conforming finite element scheme* for approximating solutions to the generalized von Kármán equations. It thus serves as a natural complement to Ciarlet, Gratie and Sabu[10], where the *existence* of such solutions was established.

To begin with, we briefly review in Sec. 2 the genesis and various mathematical features of the generalized von Kármán equations, before describing in Sec. 3 a natural finite element method for approximating these equations. In particular, we recall in Theorem 2.1 why solving these equations amounts to finding a solution $\xi \in V(\omega)$ to a "cubic" operator equation of the form

$$\tilde{C}(\xi) + \xi - \tilde{B}(\chi, \xi) - F = 0 \text{ in } V(\omega),$$

where

$$V(\omega) := \{\eta \in H^2(\omega);\ \eta = \partial_\nu \eta = 0 \ \text{ on } \ \gamma_1\}.$$

Here, ω is a bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set $\bar{\omega}$ representing the middle surface of the plate, and γ_1 is a portion of γ satisfying $0 < length\,\gamma_1 < length\,\gamma$. The nonlinearity in this equation lies in the operator $\tilde{C} : V(\omega) \to V(\omega)$, which is "cubic", in the sense that $\tilde{C}(\alpha\eta) = \alpha^3\tilde{C}(\eta)$

for all $\alpha \in \mathbb{R}$ and $\eta \in V(\omega)$. Otherwise, $\tilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega)$ is a bilinear mapping and $\chi \in H^2(\omega)$ and $F \in V(\omega)$ are given functions. The classical von Kármán equations correspond to the case where $\gamma_1 = \gamma$.

The above cubic operator equation generalizes an operator equation originally introduced by Berger[2] and Berger and Fife[3] in the case where $\gamma_1 = \gamma$ and $F = 0$, and also used by Naumann[20] when $\gamma_1 = \emptyset$ and $\chi = 0$. However, this equation now displays two features that were not encountered in these special cases or in the classical von Kármán equations. First, *its leading term "loses its strict positivity"*, in the sense that, for an *ad hoc* inner-product $((\cdot,\cdot))$ on the space $V(\omega)$, the inequality $((\tilde{C}(\eta),\eta)) \geq 0$ holds for all $\eta \in V(\omega)$, but $((\tilde{C}(\eta),\eta)) = 0$ no longer necessarily implies that $\eta = 0$. Second, *the bilinear form*

$$(\xi,\eta) \in V(\omega) \times V(\omega) \to ((\tilde{B}(\chi,\xi),\ \eta))$$

is no longer necessarily symmetric, a property that prevents solving the operator equation by way of finding the stationary points of an associated functional.

These two reasons, which are discussed with more details in Sec. 5, basically preclude the usage of those techniques that were successfully employed for solving the operator equation corresponding to the classical von Kármán equations, such as the minimization of an associated functional as in Sec. 2.2 of Ciarlet and Rabier[11] or in Theorem 5.8-3 of Ciarlet[6], or the topological degree as in Goeleven, Nguyen and Théra[14], or the recourse to pseudo-monotone operators as in Gratie[15]. The same reasons also preclude the usage of finite element methods proposed by Kesavan[18].

To overcome these difficulties, we make instead an essential use of a crucial *compactness method* due to J. L. Lions[19], which itself relies on *Brouwer's fixed point theorem.* More specifically, we establish in Sec. 4 (see Theorem 4.1) the convergence of the finite element solutions by combining J. L. Lions' method with various functional analytic tools and specific properties of the discrete mappings approximating the "continuous" mappings $\tilde{C}$ and $\tilde{B}$. Note that, interestingly, our convergence proof also provides as a by-product another proof of the *existence* of solutions to the operator equations.

The results of the present paper were announced in Ref. 9.

2. The generalized von Kármán equations

Greek indices, corresponding to the coordinates in the "horizontal" plane, vary in {1,2} and Latin indices vary in {1,2,3}, except if they are used for indexing sequences. The summation convention with respect to repeated indices is systematically used.

Let there be given a bounded, connected, simply-connected, open subset ω of the "horizontal" plane $\mathbb{R}^2$ with a sufficiently smooth boundary γ, the set ω being locally on a single side of γ. Without loss of generality, it is assumed that the origin of $\mathbb{R}^2$ belongs to γ. Let γ_1 and γ_2 be two disjoint relatively open subsets of γ such that *length* $\gamma_1 > 0$, *length* $\gamma_2 > 0$, and *length* $(\gamma - \{\gamma_1 \bigcup \gamma_2\}) = 0$. Let $y = (y_\alpha)$

denote a generic point in $\bar{\omega}$, and let $\partial_\alpha = \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} = \partial^2/\partial y_\alpha \partial y_\beta$. Let (ν_α) denote the unit outer normal vector along γ, let (τ_α) denote the unit tangent vector along γ defined by $\tau_1 = -\nu_2, \tau_2 = \nu_1$, and finally, let $\partial_\nu = \nu_\alpha \partial_\alpha$ and $\partial_\tau = \tau_\alpha \partial_\alpha$ denote the outer normal and tangential derivative operators along γ.

Consider a *nonlinearly elastic plate,* with *middle surface* $\bar{\omega}$ and *thickness* 2ε, whose constituting material is *homogeneous* and *isotropic*, and whose reference configuration $\bar{\omega} \times [-\varepsilon, \varepsilon]$ is a *natural state.* The behavior of this material is thus governed by its two *Lamé constants* $\lambda > 0$ and $\mu > 0$.

The plate is subjected to *vertical body forces* with density $(0, 0, f_3)$ in its interior $\omega \times]-\varepsilon, \varepsilon[$ and to *vertical surface forces* with density $(0, 0, g_3)$ on its upper and lower faces $\omega \times \{+\varepsilon\}$ and $\omega \times \{-\varepsilon\}$, where $f_3 \in L^2(\omega \times]-\varepsilon, \varepsilon[)$ and $g_3 \in L^2(\omega \times \{-\varepsilon, \varepsilon\})$. On the portion $\gamma_1 \times [-\varepsilon, \varepsilon]$ of its lateral face, the plate is subjected to *horizontal forces "of von Kármán's type"*, of the form introduced by Ciarlet[5]; this means that only the density (k_α), where $k_\alpha \in L^2(\gamma_1)$, of their *resultant after integration across the thickness of the plate* is known along γ_1 and that, accordingly, the admissible displacements along $\gamma_1 \times [-\varepsilon, \varepsilon]$ are those whose horizontal components are independent of the vertical variable and whose vertical component vanishes. Finally, the plate is subjected to a boundary condition of *free edge* on the remaining portion $\gamma_2 \times [-\varepsilon, \varepsilon]$ of its lateral face.

As shown in Ciarlet and Gratie[7], the leading term of a *formal* asymptotic expansion of the three-dimensional displacement field inside the plate, with the thickness as the "small" parameter, can be fully computed from the solution of a *two-dimensional "displacement" boundary value problem* posed over ω, i.e., whose unknowns are the three components of the "limit" displacement field of the middle surface $\bar{\omega}$ (it is likely that the "gamma-convergence approach" successfully used by Friesecke, James and Müller[12,13], for modeling the two-dimensional equations of a clamped plate, would likewise fully justify this two-dimensional boundary value problem, although this assertion is yet to be rigorously substantiated).

The main result in Ref. 7 then consisted in showing that, under the assumptions that the set ω is simply connected and that its boundary γ is smooth enough, there is a one-to-one correspondence between the smooth solutions of this boundary value problem and those of another boundary value problem, which takes the form of the following *generalized von Kármán equations*:

$$\begin{aligned}
-\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 \xi) &= [\phi,\ \xi] + f \text{ in } \omega,\\
\Delta^2 \phi &= -\ [\xi,\ \xi] \text{ in } \omega,\\
\xi = \partial_\nu \xi &= 0 \quad \text{on } \gamma_1,\\
m_{\alpha\beta}(\nabla^2 \xi)\nu_\alpha \nu_\beta &= 0 \quad \text{on } \gamma_2,\\
\partial_\alpha m_{\alpha\beta}(\nabla^2 \xi)\nu_\beta + \partial_\tau (m_{\alpha\beta}(\nabla^2 \xi)\nu_\alpha \tau_\beta) &= 0 \quad \text{on } \gamma_2,\\
\phi = \phi_0 \text{ and } \partial_\nu \phi &= \phi_1 \quad \text{on } \gamma.
\end{aligned}$$

The *two unknowns* $\xi : \bar{\omega} \to \mathbb{R}$ and $\phi : \bar{\omega} \to \mathbb{R}$ and the various notations appearing in these equations have the following significance: The function $E^{-1/2}\xi : \bar{\omega} \to \mathbb{R}$

is the *vertical component of the displacement field of the middle surface* $\bar{\omega}$ *of the plate*, the constant $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ standing for the Young modulus of its constituting material. The function $\varepsilon^{-2}\phi : \bar{\omega} \to \mathbb{R}$ is the *Airy function*, from which the horizontal components of the displacement can be in turn determined (see *ibid.*). The *Monge-Ampère form* $[\cdot,\cdot]$ is defined for smooth enough functions $\phi : \bar{\omega} \to \mathbb{R}$ and $\xi : \bar{\omega} \to \mathbb{R}$ by

$$[\phi, \xi] = \partial_{11}\phi\partial_{22}\xi + \partial_{22}\phi\partial_{11}\xi - 2\partial_{12}\phi\partial_{12}\xi.$$

The given function $f \in L^2(\omega)$ is defined by (x_3 designates the vertical variable)

$$f := \varepsilon^4 E^{-1/2}\left\{ \int_{-\varepsilon}^{\varepsilon} f_3 dx_3 + g_3(.,\varepsilon) + g_3(.,-\varepsilon) \right\}.$$

The functions $m_{\alpha\beta}(\nabla^2\xi)$ are defined as

$$m_{\alpha\beta}(\nabla^2\xi) := -\frac{1}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\xi, \text{ where } a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}).$$

This means that the functions $\varepsilon^4 E^{-1/2} m_{\alpha\beta}(\nabla^2\xi)$ represent the *bending moments* inside the plate. Finally, the functions $\phi_0 \in H^{3/2}(\gamma)$ and $\phi_1 \in H^{1/2}(\gamma)$ are defined in terms of the given functions $k_\alpha \in L^2(\gamma_1)$ by

$$\phi_0(y) := -y_1 \int_{\gamma(y)} \tilde{k}_2(x)\mathrm{d}\gamma(x) + y_2 \int_{\gamma(y)} \tilde{k}_1(x)\mathrm{d}\gamma(x) + \int_{\gamma(y)} (x_1\tilde{k}_2(x) - x_2\tilde{k}_1(x))\mathrm{d}\gamma(x),$$
$$\phi_1(y) := -\nu_1(y) \int_{\gamma(y)} \tilde{k}_2(x)\mathrm{d}\gamma(x) + \nu_2(y) \int_{\gamma(y)} \tilde{k}_1(x)\mathrm{d}\gamma(x),$$

where $\gamma(y)$ designates the oriented arc from $0 \in \gamma$ to $y \in \gamma$ and the functions $\tilde{k}_\alpha \in L^2(\gamma)$ are defined by $\tilde{k}_\alpha = k_\alpha$ on γ_1 and $\tilde{k}_\alpha = 0$ on γ_2 (in these integrals, $x = (x_1, x_2) \in \gamma$ is the integration variable). Naturally, we need to assume that the functions $\tilde{k}_\alpha \in L^2(\gamma)$ satisfy the *compatibility relations*

$$\int_\gamma \tilde{k}_1 \mathrm{d}\gamma = \int_\gamma \tilde{k}_2 \mathrm{d}\gamma = \int_\gamma (x_1\tilde{k}_2 - x_2\tilde{k}_1)\mathrm{d}\gamma = 0.$$

For, as is easily verified (see, e.g., the proof of Theorem 5.6-1 in Ref. 6), these compatibility relations guarantee that the functions ϕ_0 and ϕ_1 are indeed well-defined as functions in the spaces $H^{3/2}(\gamma)$ and $H^{1/2}(\gamma)$, respectively. The simple-connectedness of ω is also used here.

Note that these "generalized" von Kármán equations indeed generalize the *"classical" von Kármán equations* that correspond to the case where $\gamma_1 = \gamma$. Detailed treatments of these classical equations are found in Ref. 6 and Ref. 11.

Once the generalized von Kármán equations are derived under the key assumption that ω is simply-connected and γ is smooth and that their solutions (ξ, ϕ) are smooth, they can be studied for their own sake, in particular regarding the existence of less smooth solutions when the set ω is not necessarily simply-connected, and when the boundary γ is only Lipschitz-continuous in the sense of Nečas[21] or Adams[1].

One can establish the following existence theorem (see Ciarlet, Gratie and Sabu[10]):

Theorem 2.1. *Let ω be a bounded, connected, open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, and let $f \in L^2(\omega)$ and $(\phi_0, \phi_1) \in H^{3/2}(\gamma) \times H^{1/2}(\gamma)$ be given functions. Then, if the norm $\|(\phi_0, \phi_1\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ is small enough, the generalized von Kármán equations have at least one weak solution $(\xi, \phi) \in H^2(\omega) \times H^2(\omega)$.*

Elements from the proof. We only recapitulate here those parts of the existence proof from Ref. 10 that will be needed in the sequel; for the other parts of this proof, see *ibid.*

(i) Define the *bilinear mapping:*

$$B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega)$$

as follows: Given $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution of the variational equations (note that $[\xi, \eta] \in L^1(\omega)$, so that their right-hand side makes sense):

$$\int_\omega \Delta B(\xi, \eta) \Delta\theta d\omega = \int_\omega [\xi, \eta]\, \theta d\omega \ \text{ for all } \theta \in H_0^2(\omega).$$

(ii) Define another *bilinear mapping*

$$\tilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega) := \{\eta \in H^2(\omega);\ \eta = \partial_\nu \eta = 0 \ \text{ on } \ \gamma_1\}$$

as follows: Given $(\phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\tilde{B}(\phi, \xi) \in V(\omega)$ is the unique solution of the variational equations:

$$((\tilde{B}(\phi, \xi), \eta)) = \int_\omega [\phi, \xi]\, \eta d\omega \ \text{ for all } \eta \in V(\omega),$$

where the inner-product $((\cdot, \cdot))$ is defined by

$$((\zeta, \eta)) = \frac{1}{3} \int_\omega a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \zeta \partial_{\alpha\beta} \eta d\omega.$$

Note that the norm $\|\cdot\|$ associated with the inner product $((\cdot, \cdot))$ is equivalent to the norm $\|\cdot\|_{H^2(\omega)}$ over the space $V(\omega)$.

(iii) Let $\chi \in H^2(\omega)$ be the unique solution of the variational equations

$$\int_\omega \Delta\chi \Delta\theta d\omega = 0 \ \text{ for all } \theta \in H_0^2(\omega)$$

that also satisfies $\chi = \phi_0$ and $\partial_\nu \chi = \phi_1$ on γ.

(iv) Finally, let $F \in V(\omega)$ denote the unique solution of the variational equations

$$((F, \eta)) = \frac{1}{3} \int_\omega f \eta d\omega \ \text{ for all } \eta \in V(\omega).$$

(v) Then *finding a weak solution* (ξ, ϕ) *of the generalized von Kármán equations amounts to finding* $\xi \in V(\omega)$ *that satisfies the operator equation:*

$$\tilde{C}(\xi) + \xi - \tilde{B}(\chi, \xi) - F = 0 \text{ in } V(\omega),$$

where the *nonlinear mapping*

$$\tilde{C} : V(\omega) \to V(\omega)$$

is defined by

$$\tilde{C}(\eta) := \tilde{B}(B(\eta, \eta), \eta), \quad \text{for all } \eta \in V(\omega),$$

the unknown $\phi \in H^2(\omega)$ being then given by

$$\phi = \chi - B(\xi, \xi).$$

□

Naturally, finding the solution ξ of the above operator equation is equivalent to solving the following *variational problem*: Find ξ such that

(P) $\quad \xi \in V(\omega)$ and $((\tilde{C}(\xi) + \xi - \tilde{B}(\chi, \xi) - F, \eta)) = 0$ for all $\eta \in V(\omega)$.

Note that the nonlinear mapping $\tilde{C}$ is "*cubic*", in the sense that $\tilde{C}(\alpha\eta) = \alpha^3 \tilde{C}(\eta)$ for all $\alpha \in \mathbb{R}$ and $\eta \in V(\omega)$.

3. The discrete problem

In order to avoid technicalities due to possibly curved portions of the boundary γ, we henceforth assume that γ is a polygon, so that $\bar{\omega}$ can be exactly covered by a regular family of triangulations. Let

$$W_h \subset H^2(\omega), \; V_h \subset V(\omega), \; V_{0h} \subset H_0^2(\omega)$$

be *standard conforming finite element spaces* associated with such a family, i.e., that satisfy the minimal conditions of Theorem 6.1-7 in Ref. 4. As usual, the parameter h denotes the greatest diameter of all the finite elements found in a given triangulation. For each $h > 0$, the discrete problem is then defined through the following stages, which simply mimic those that lead to the operator equation satisfied by $\xi \in V(\omega)$ (see parts (i) to (v) in the "elements of the proof" of Theorem 2.1):

(i) Define the *bilinear mapping*

$$B_h : H^2(\omega) \times H^2(\omega) \to V_{0h}$$

as follows: Given $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B_h(\xi, \eta) \in V_{0h}$ is the unique solution of the variational equations:

$$\int_\omega \Delta B_h(\xi, \eta) \Delta \theta_h d\omega = \int_\omega [\xi, \eta]\, \theta_h d\omega \quad \text{for all } \theta_h \in V_{0h}.$$

Hence, for $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$ *fixed,*

$$\|B_h(\xi, \eta) - B(\xi, \eta)\|_{H^2(\omega)} \to 0 \text{ as } h \to 0.$$

(ii) Define another *bilinear mapping*:

$$\tilde{B}_h : H^2(\omega) \times H^2(\omega) \to V_h$$

as follows: Given $(\phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\tilde{B}_h(\phi, \xi) \in V_h$ is the unique solution of the variational equations

$$((\tilde{B}_h(\phi, \xi), \eta_h)) = \int_\omega [\phi, \xi]\, \eta_h d\omega \quad \text{for all } \eta_h \in V_h.$$

Hence, for $(\phi, \xi) \in H^2(\omega) \times H^2(\omega)$ *fixed*,

$$\left\|\tilde{B}_h(\phi, \xi) - \tilde{B}(\phi, \xi)\right\|_{H^2(\omega)} \to 0 \text{ as } h \to 0.$$

(iii) Let $\chi_h \in W_h$ be a standard finite element approximation of $\chi \in H^2(\omega)$, which therefore satisfies $\|\chi_h - \chi\|_{H^2(\omega)} \to 0$ as $h \to 0$.

(iv) Finally, let $F_h \in V_h$ be the unique solution of the variational equations

$$((F_h,\ \eta_h)) = \int_\omega f \eta_h d\omega \quad \text{for all } \eta_h \in V_h(\omega),$$

which therefore satisfies

$$\|F_h - F\|_{H^2(\omega)} \to 0 \text{ as } h \to 0.$$

(v) Then the *discrete problem* consists in finding $(\xi_h, \phi_h) \in V_h \times W_h$ in two stages: *First*, $\xi_h \in V_h$ is found by solving the *discrete operator equation*:

$$\tilde{C}_h(\xi_h) + \xi_h - \tilde{B}_h(\chi_h, \xi_h) - F_h = 0 \text{ in } V_h,$$

where the *discrete "cubic" mapping* $\tilde{C}_h : V_h \to V_h$ is defined by

$$\tilde{C}_h(\eta_h) := \tilde{B}_h(B_h(\eta_h, \eta_h), \eta_h) \text{ for all } \eta_h \in V_h.$$

Note that finding ξ_h is clearly equivalent to solving the following *discrete variational problem* (which is shown to have at least one solution in Theorem 4.1 below): Find ξ_h such that

$$(\mathrm{P}_h) \quad \xi_h \in V_h \text{ and } ((\tilde{C}_h(\xi_h) + \xi_h - \tilde{B}_h(\chi_h, \xi_h) - F_h, \eta_h)) = 0 \text{ for all } \eta_h \in V_h.$$

Second, $\phi_h \in W_h$ is given by

$$\phi_h = \chi_h - B_h(\xi_h, \xi_h).$$

4. Convergence

The following theorem establishes the *convergence* of the finite element method described in Sec. 3. Interestingly, the same theorem automatically provides in addition the *existence* of a weak solution to the generalized von Kármán equations (which otherwise can be established by a direct proof; see Ref. 10). Strong and weak convergences are denoted $\to$ and $\rightharpoonup$ respectively. All convergences are meant to hold as h approaches zero.

Theorem 4.1. *Assume that the norm* $\|(\phi_0, \phi_1)\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ *is small enough. Then there exists a constant* M *such that, for each* $h > 0$, *the discrete variational problem* (P_h) (cf. Sec. 2) *has at least one solution* $\xi_h \in V_h$ *that satisfies* $\|\xi_h\| \leq M$. *Let* $(\xi_h)_{h>0}$ *be any subsequence that weakly converges in* $H^2(\omega)$ *and let the associated subsequence* $(\phi_h)_{h>0}$ *be defined by* $\phi_h = \chi_h - B_h(\xi_h, \xi_h)$. *Then*

$$(\xi_h, \phi_h) \to (\xi, \phi) \ \ in\ H^2(\omega) \times H^2(\omega),$$

where (ξ, ϕ) *is a weak solution of the generalized von Kármán equations.*

Proof.

For clarity, the proof is broken into several parts, numbered (i) to (viii). We begin by a property of the Monge-Ampère form that will be crucially needed in parts (ii) and (iii). Although its proof is known (see, e.g., Theorem 5.8-2 in Ref. 6), it is nevertheless reproduced here for the sake of completeness.

In what follows, c_1, c_2, etc., designate various strictly positive constants that may depend on ω, but are otherwise independent of h.

(i) *The trilinear form*

$$T : (\xi, \eta, \chi) \in H^2(\omega) \times H^2(\omega) \times H^2(\omega) \to \int_\omega [\xi, \eta]\, \chi d\omega$$

is continuous, i.e,

$$\left| \int_\omega [\xi, \eta]\, \chi d\omega \right| \leq c_1 \|\xi\|_{H^2(\omega)} \|\eta\|_{H^2(\omega)} \|\chi\|_{H^2(\omega)}$$

for all $(\xi, \eta, \chi) \in H^2(\omega) \times H^2(\omega) \times H^2(\omega)$. *Moreover,* T *becomes a symmetric form if at least one of its three arguments is in* $H_0^2(\omega)$.

We clearly have

$$\left| \int_\omega [\xi, \eta]\, \chi d\omega \right| \leq \|[\xi, \eta]\|_{L^1(\omega)} \|\chi\|_{C^0(\bar{\omega})} \cdot$$

Hence the continuity of the trilinear form T follows from the definition of $[\xi, \eta]$ and from the continuous imbedding of the space $H^2(\omega)$ into the space $C^0(\bar{\omega})$. Let the functions ξ, η, and χ be in $C^\infty(\bar{\omega})$; we may then write

$$\begin{aligned}
\int_\omega [\xi, \eta]\, \chi d\omega &= \int_\omega (\chi \partial_{11}\xi \partial_{22}\eta - \chi\partial_{12}\xi\partial_{12}\eta) d\omega + \int_\omega (\chi\partial_{22}\xi\partial_{11}\eta - \chi\partial_{12}\xi\partial_{12}\eta) d\omega \\
&= \int_\omega \partial_2(\chi\partial_{11}\xi\partial_2\eta - \chi\partial_{12}\xi\partial_1\eta) d\omega - \int_\omega \partial_2\eta\partial_2(\chi\partial_{11}\xi) d\omega + \int_\omega \partial_1\eta\partial_2(\chi\partial_{12}\xi) d\omega \\
&+ \int_\omega \partial_1(\chi\partial_{22}\xi\partial_1\eta - \chi\partial_{12}\xi\partial_2\eta) d\omega - \int_\omega \partial_1\eta\partial_1(\chi\partial_{22}\xi) d\omega + \int_\omega \partial_2\eta\partial_1(\chi\partial_{12}\xi) d\omega.
\end{aligned}$$

If *at least one* of the three functions ξ, η, and χ, is in $\mathcal{D}(\omega)$, the integrals $\int\limits_\omega \partial_\alpha(...)d\omega$ vanish and we are left with

$$\int\limits_\omega [\xi,\eta]\,\chi d\omega = \int\limits_\omega \partial_{12}\xi(\partial_1\eta\partial_2\chi + \partial_2\eta\partial_1\chi)d\omega - \int\limits_\omega (\partial_{11}\xi\partial_2\eta\partial_2\chi + \partial_{22}\xi\partial_1\eta\partial_1\chi)d\omega.$$

Since $\overline{\mathcal{C}^\infty(\bar{\omega})} = H^2(\omega)$ and $\overline{\mathcal{D}(\omega)} = H_0^2(\omega)$, and since both sides are continuous trilinear forms with respect to $\|\cdot\|_{H^2(\omega)}$ (recall that the space $H^2(\omega)$ is also continuously imbedded in the space $W^{1,4}(\omega)$), this relation remains valid if the functions ξ, η, and χ belong to $H^2(\omega)$, one of them being in $H_0^2(\omega)$; hence the trilinear form becomes symmetric in this case: The left-hand side is unaltered if ξ and η are exchanged and likewise, the right-hand side is unaltered if η and χ are exchanged.

(ii) *The discrete cubic mapping* $\tilde{C}_h : V_h \to V_h$ *satisfies*

$$((\tilde{C}_h(\eta_h), \eta_h)) \geq 0 \;\text{ for all } \eta_h \in V_h.$$

To see this, let η_h be any function in the space V_h. Then, by definition of the mapping $\tilde{B}_h$,

$$((\tilde{C}_h(\eta_h), \eta_h)) = ((\tilde{B}_h(B_h(\eta_h,\eta_h),\eta_h),\eta_h)) = \int\limits_\omega [B_h(\eta_h,\eta_h),\eta_h]\,\eta_h d\omega.$$

But, since $B_h(\eta_h,\eta_h) \in V_{0h} \subset H_0^2(\omega)$, we may also write

$$\int\limits_\omega [B_h(\eta_h,\eta_h),\eta_h]\,\eta_h d\omega = \int\limits_\omega [\eta_h,\eta_h]\,B_h(\eta_h,\eta_h)d\omega = \int\limits_\omega |\Delta B_h(\eta_h,\eta_h)|^2\,d\omega$$

by definition of the mapping B_h, thus establishing the announced inequality.

(iii) *Let* $\chi_h \in W_h, \xi_h \in W_h$, *and* $\eta_h \in W_h$ *be such that*

$$\chi_h \to \chi \text{ in } H^2(\omega),\; \xi_h \rightharpoonup \xi \text{ in } H^2(\omega),\; \eta_h \rightharpoonup \eta \text{ in } H^2(\omega).$$

Then

$$((\tilde{B}_h(\chi_h,\xi_h),\eta_h)) \to ((\tilde{B}(\chi,\xi),\eta)).$$

By definition of the mappings $\tilde{B}_h$ and $\tilde{B}$, we have

$$\begin{aligned}((\tilde{B}_h(\chi_h,\xi_h),\eta_h)) - ((\tilde{B}(\chi,\xi),\eta)) &= \int\limits_\omega [\chi_h,\xi_h]\eta_h d\omega - \int\limits_\omega [\chi,\xi]\eta d\omega \\ &= \int\limits_\omega [\chi_h-\chi,\xi_h]\eta_h d\omega + \int\limits_\omega [\chi,\xi_h](\eta_h-\eta)d\omega + \int\limits_\omega [\chi,\xi_h-\xi]\eta d\omega.\end{aligned}$$

Let us examine each term separately. Since a weakly convergent sequence in $H^2(\omega)$ is bounded and the injection of $H^2(\omega)$ into $\mathcal{C}^0(\bar{\omega})$ is compact, the inequalities

$$\int\limits_\omega [\chi_h-\chi,\xi_h]\eta_h d\omega \leq c_1 \|\chi_h-\chi\|_{H^2(\omega)}\,\|\xi_h\|_{H^2(\omega)}\,\|\eta_h\|_{H^2(\omega)},$$

and

$$\int_\omega [\chi, \xi_h] \, (\eta_h - \eta) d\omega \leq \|[\chi, \xi_h]\|_{L^1(\omega)} \|\eta_h - \eta\|_{C^0(\bar{\omega})}$$
$$\leq c_2 \|\chi\|_{H^2(\omega)} \|\xi_h\|_{H^2(\omega)} \|\eta_h - \eta\|_{C^0(\bar{\omega})},$$

imply that the first and second terms approach zero as $h \to 0$ (recall that $\chi_h \to \chi$ in $H^2(\omega)$ by assumption). Since the functions $\eta \partial_{\alpha\beta}\chi$ belong to $L^2(\omega)$ and since $\partial_{\sigma\tau}\xi_h \rightharpoonup \partial_{\sigma\tau}\xi$ in $L^2(\omega)$, the relation

$$\int_\omega [\chi, \xi_h - \xi] \, \eta d\omega = \int_\omega \{\eta \partial_{11}\chi \partial_{22}\xi_h + \eta \partial_{22}\chi \partial_{11}\xi_h - 2\eta \partial_{12}\chi \partial_{12}\xi_h\} d\omega - \int_\omega [\chi, \xi] \, \eta d\omega$$

implies that the third term likewise approaches zero as $h \to 0$.

(ii) *Let* $\xi_h \in W_h$ *be such that*

$$\xi_h \rightharpoonup \xi \text{ in } H^2(\omega).$$

Then

$$B_h(\xi_h, \xi_h) \to B(\xi, \xi) \text{ in } H_0^2(\omega).$$

Let $\zeta_h := B_h(\xi_h, \xi_h) - B_h(\xi, \xi_h)$ and $\psi_h := B_h(\xi, \xi_h) - B_h(\xi, \xi)$, so that

$$\|B_h(\xi_h, \xi_h) - B(\xi, \xi)\|_{H^2(\omega)} \leq \|\zeta_h\|_{H^2(\omega)} + \|\psi_h\|_{H^2(\omega)}.$$

The symmetry property established in (i), the inclusion $V_{0h} \subset H_0^2(\omega)$, and the definition of the mapping B_h together imply that

$$\int_\omega \Delta\zeta_h \Delta\theta_h d\omega = \int_\omega [\xi_h - \xi, \xi_h] \, \theta_h d\omega = \int_\omega [\theta_h, \xi_h] \, (\xi_h - \xi) d\omega$$

for all $\theta_h \in V_{0h}$. Letting $\theta_h = \zeta_h$ in these equations thus gives

$$\|\zeta_h\|^2_{H^2(\omega)} \leq c_3 \|\Delta\zeta_h\|^2_{L^2(\omega)} \leq c_3 \|[\zeta_h, \xi_h]\|_{L^1(\omega)} \|\xi_h - \xi\|_{C_0(\bar{\omega})}$$
$$\leq c_2 c_3 \|\zeta_h\|_{H^2(\omega)} \|\xi_h\|_{H^2(\omega)} \|\xi_h - \xi\|_{C_0(\bar{\omega})}.$$

Hence $\zeta_h \to 0$ in $H_0^2(\omega)$, again because a weakly convergent sequence in $H^2(\omega)$ is bounded and the injection of $H_0^2(\omega)$ in $C^0(\bar{\omega})$ is compact. The same kind of argument, now applied to the equations

$$\int_\omega \Delta\psi_h \Delta\theta_h d\omega = \int_\omega [\xi, \xi_h - \xi] \, \theta_h d\omega = \int_\omega [\xi, \theta_h] \, (\xi_h - \xi) d\omega$$

for all $\theta_h \in V_{0h}$, likewise shows that $\psi_h \to 0$ in $H_0^2(\omega)$. Hence the assertion is proved.

(v) *If the norm* $\|(\phi_0, \phi_1)\|_{H^{3/2}(\gamma) \times H^{1/2}(\gamma)}$ *is small enough, there exists a constant* M *independent of* h *such that problem* (P_h) *has at least one solution* ξ_h *that satisfies* $\|\xi_h\| \leq M$.

This part of the proof is inspired by a crucial *compactness method of J. L. Lions* (see Theorem 4.3, Chap. 1 of Lions[19]). Let w_i^h, $1 \leq i \leq d(h)$, be a basis of V_h

that is orthonormal with respect to the inner product $((\cdot,\cdot))$ and let the mapping $\mathbf{G}^h = (G_i^h) : \mathbb{R}^{d(h)} \to \mathbb{R}^{d(h)}$ be defined for each $\mathbf{X} = (X_i) \in \mathbb{R}^{d(h)}$ by

$$G_i^h(\mathbf{X}) := ((\tilde{C}_h(\eta_h(\mathbf{X})) + \eta_h(\mathbf{X}) - \tilde{B}_h(\chi_h, \eta_h(\mathbf{X})) - F_h, w_i^h)),\ 1 \le i \le d(h),$$

where

$$\eta_h(\mathbf{X}) := \sum_{i=1}^{d(h)} X_i w_i^h.$$

First, we note that the mapping $\mathbf{G}^h : \mathbb{R}^{d(h)} \to \mathbb{R}^{d(h)}$ defined in this fashion is continuous, since linear and bilinear mappings between finite-dimensional spaces are continuous. Next, let $|\cdot|$ and $<\cdot,\cdot>$ denote the Euclidean norm and inner product in $\mathbb{R}^{d(h)}$ and let $\mathbf{X}$ be any vector in $\mathbb{R}^{d(h)}$. Then

$$\begin{aligned}< \mathbf{G}^h(\mathbf{X}), \mathbf{X} > &= ((\tilde{C}_h(\eta_h(\mathbf{X})), \eta_h(\mathbf{X}))) + \|\eta_h(\mathbf{X})\|^2 \\ &\quad - ((\tilde{B}_h(\chi_h, \eta_h(\mathbf{X})), \eta_h(\mathbf{X})) - ((F_h, \eta_h(\mathbf{X}))) \\ &\ge |\mathbf{X}|^2 - \left(\left\| \tilde{B}_h(\chi_h, \eta_h(\mathbf{X})) \right\| + c_4 \|f\|_{L^2(\omega)} \right) |\mathbf{X}|,\end{aligned}$$

since the first term is ≥ 0 by (ii) and $\|\eta_h(\mathbf{X})\| = |\mathbf{X}|$. Besides,

$$\begin{aligned}\left\| \tilde{B}_h(\chi_h, \eta_h(\mathbf{X})) \right\| &= \sup \left\{ \frac{1}{\|\theta_h\|} \int_\omega [\chi_h, \eta_h(\mathbf{X})]\, \theta_h d\omega \ ;\ \theta_h \in V_h,\ \theta_h \ne 0 \right\} \\ &\le c_5 \|\chi\|_{H^2(\omega)} |\mathbf{X}|,\end{aligned}$$

since $\|\chi_h\|_{H^2(\omega)} \le c_6 \|\chi\|_{H^2(\omega)}$. Consequently,

$$< \mathbf{G}^h(\mathbf{X}), \mathbf{X} > \ge (1 - c_5 \|\chi\|_{H^2(\omega)}) |\mathbf{X}|^2 - c_4 \|f\|_{L^2(\omega)} |\mathbf{X}| \quad \text{for all } \mathbf{X} \in \mathbb{R}^{d(h)}.$$

Assume that the norm $\|(\phi_0, \phi_1)\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ is small enough, in the sense that $\|\chi\|_{H^2(\omega)} < c_5^{-1}$. Then choose $M = M\left(\|\chi\|_{H^2(\omega)}, \|f\|_{L^2(\omega)}\right) > 0$ such that

$$(1 - c_5 \|\chi\|_{H^2(\omega)}) M^2 - c_4 \|f\|_{L^2(\omega)} M \ge 0.$$

As a result, the continuous mapping $\mathbf{G}^h : \mathbb{R}^{d(h)} \to \mathbb{R}^{d(h)}$ satisfies

$$< \mathbf{G}^h(\mathbf{X}), \mathbf{X} > \ge 0 \quad \text{for all } \mathbf{X} \in \mathbb{R}^{d(h)} \text{ such that } |X| = M.$$

By a well-known corollary to the *Brouwer fixed point theorem* (see, e.g., Lemma 4.3, Chap. 1, of Ref. 19) there thus exists at least one vector $\mathbf{X} \in \mathbb{R}^{d(h)}$ such that

$$\mathbf{G}^h(\mathbf{X}) = 0 \text{ and } |\mathbf{X}| \le M.$$

Equivalently, there thus exists at least one solution $\xi_h := \sum_{i=1}^{d(h)} X_i w_i^h$ to problem (P_h) that satisfies $\|\xi_h\| \le M$.

(vi) *Let $(\xi_h)_{h>0}$ be any subsequence of the sequence found in* (v) *that satisfies*

$$\xi_h \rightharpoonup \xi \text{ in } H^2(\omega).$$

Then ξ is a solution of the variational problem (P).

Given any $\eta \in V(\omega)$ let $\eta_h \in V_h$ be such that $\eta_h \to \eta$ in $H^2(\omega)$. Hence, for any $h > 0$,

$$((\tilde{C}_h(\xi_h) + \xi_h - \tilde{B}_h(\chi_h, \xi_h) - F_h, \eta_h)) = 0.$$

First, it is clear that $((\xi_h - F_h, \eta_h)) \to ((\xi - F, \eta))$. Next, $((\tilde{B}_h(\chi_h, \xi_h), \eta_h)) \to ((B(\chi, \xi), \eta))$ by (iii). Finally, part (iii) again shows that

$$((\tilde{C}_h(\xi_h), \eta_h)) = ((\tilde{B}_h(B_h(\xi_h, \xi_h), \xi_h), \eta_h)) \to ((\tilde{B}(B(\xi, \xi), \xi), \eta)) = ((\tilde{C}(\xi), \eta)),$$

since $B_h(\xi_h, \xi_h) \to B(\xi, \xi)$ in $H_0^2(\omega)$ by (iv).

(vii) *The weakly convergent subsequence $(\xi_h)_{h>0}$ considered in* (vi) *is in fact strongly convergent in $H^2(\omega)$.*

Letting $\eta_h = \xi_h$ in the variational equations of (P_h) gives

$$((\tilde{C}_h(\xi_h), \xi_h)) + \|\xi_h\|^2 - ((\tilde{B}_h(\chi_h, \xi_h), \xi_h)) - ((F(\xi_h), \xi_h)) = 0.$$

Then

$$((\tilde{C}_h(\xi_h), \xi_h)) = ((\tilde{B}_h(B_h(\xi_h, \xi_h), \xi_h), \xi_h) \to ((\tilde{B}(B(\xi, \xi), \xi), \xi) = ((\tilde{C}(\xi), \xi))$$

by (iii), since $B_h(\xi_h, \xi_h) \to B(\xi, \xi)$ in $H_0^2(\omega)$ by (iv). Likewise,

$$((\tilde{B}_h(\chi_h, \xi_h), \xi_h)) \to ((\tilde{B}(\chi, \xi), \xi))$$

by (iii) again. Since $((F_h, \xi_h)) \to ((F, \xi))$ and ξ is a solution to the variational problem (P), we therefore conclude that $\|\xi_h\|^2 \to \|\xi\|$. Hence $\xi_h \to \xi$ in $H^2(\omega)$.

(viii) *The functions $\phi_h := \chi_h - B_h(\xi_h, \xi_h)$ strongly converge in $H^2(\omega)$ to the function $\phi = \chi - B(\xi, \xi)$.*

This property immediately follows from the assumed convergence $\chi_h \to \chi$ in $H^2(\omega)$ and part (iv). □

5. Concluding remarks

(a) The cubic operator $\tilde{C} : V(\omega) \to V(\omega)$ generalizes the cubic operator $C : H_0^2(\omega) \to H_0^2(\omega)$ defined by $C(\eta) = B(B(\eta, \eta), \eta)$ for all $\eta \in H_0^2(\omega)$ that corresponds to the classical von Kármán equations. But *the operator $\tilde{C}$ loses the "strict positivity" of the operator C*, a property which is essential for establishing the existence of a solution by means of a functional, as in Theorem 2.2-1 of Ref. 11 (see also Theorem 5.8-3 of Ref. 6). To see this, note that

$$((\tilde{C}(\eta), \eta)) = \int_\omega [\eta, \eta]\, B(\eta, \eta) d\omega = \int_\omega |\Delta B(\eta, \eta)|^2\, d\omega \geq 0$$

for all $\eta \in V(\omega)$, but also note that it is easy to exhibit nonzero functions $\eta \in V(\omega)$ that satisfy $[\eta, \eta] = 0$ when $\gamma_1 \neq \gamma$. Hence $\eta \in V(\omega)$ and $((\tilde{C}(\eta), \eta)) = 0$ does not necessarily imply that $\eta = 0$, while by contrast, $[\eta, \eta] = 0$ implies $\eta = 0$ if $\eta \in H_0^2(\omega)$ (see, e.g., Theorem 5.8-2 of Ref. 6).

(b) Another feature of the cubic operator equation associated with the generalized von Kármán equations is that, in general, the bilinear form

$$(\xi, \eta) \in V(\omega) \times V(\omega) \to ((\tilde{B}(\chi, \xi),\ \eta))$$

is no longer *symmetric*, since the number

$$((\tilde{B}(\chi, \xi),\ \eta)) = \int_\omega [\chi, \xi]\eta d\omega$$

is *not* necessarily equal to $\int_\omega [\chi, \eta]\xi d\omega$ for arbitrary functions $\xi, \eta \in V(\omega)$. As already noted (see part (i) of the above proof), such an equality holds if at least one of the three functions χ, ξ, η is in the space $H_0^2(\omega)$, a condition not satisfied here. This second observation again prevents the usage of an associated functional as a means to obtain a solution to the operator equation as that of a *minimization problem* (as in the case $\gamma_1 = \gamma$; see again Theorem 2.2-1 of Ref. 11 or Theorem 5.8-3 of Ref. 6). Interestingly, the cubic term poses no problem in this respect, since it is easily verified that, for arbitrary functions $\xi, \eta \in V(\omega)$, the Gâteaux derivative $j'(\xi)\eta$ of the functional $j : V(\omega) \to \mathbb{R}$ defined by $j(\eta) := \frac{1}{4}((\tilde{C}(\eta), \eta))$ is indeed equal to $((\tilde{C}(\xi), \eta))$.

(c) Numerically finding the discrete solutions $\xi_h \in V_h$ amounts to solving a *nonlinear system of cubic polynomial equations.* To see this, let $z_j,\ 1 \le i \le d$, be a canonical basis in the finite element space V_h, and let $\xi_h = \sum_j Y_j z_j$. Then the unknowns $Y_j, 1 \le i \le d$, satisfy the equations

$$\sum_{j,k,l} X_j X_k X_l((\tilde{B}_h(B_h(z_j, z_k, z_l), z_i)) \\ + \sum_j X_j((z_j - \tilde{B}_h(\chi_h, z_j), z_i)) = ((F_h, z_i)),\ 1 \le i \le d.$$

By part (v) of the above proof, this system has at least one solution, as a consequence of Brouwer's fixed point theorem. Consequently, a *continuation method* of the form proposed by Kellogg, Li and Yorke[17] can be used for finding such a solution.

(d) The analysis of the present paper can be extended to the *generalized Marguerre-von Kármán equations*, which model a *nonlinearly elastic shallow shell* subjected to boundary conditions along its lateral face that are similar to those described in Sec. 2 for a nonlinearly elastic plate. These equations have been described and justified, again as the outcome of a formal asymptotic analysis, by Gratie[16]. The existence of solutions to these equations has been established in Ciarlet and Gratie[8].

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