

Realization of spatial sparseness by deep ReLU nets with massive data

Charles K. Chui, Shao-Bo Lin, Bo Zhang, and Ding-Xuan Zhou

Abstract—The great success of deep learning poses urgent challenges for understanding its working mechanism and rationality. The depth, structure, and massive size of the data are recognized to be three key ingredients for deep learning. Most of the recent theoretical studies for deep learning focus on the necessity and advantages of depth and structures of neural networks. In this paper, we aim at rigorous verification of the importance of massive data in embodying the out-performance of deep learning. In particular, we prove that the massiveness of data is necessary in realizing the spatial sparseness and deep nets are crucial tools to make full use of massive data in such an application. All these findings present reasons why deep learning achieves great success in the era of big data, though deep nets and numerous network structures have been proposed at least twenty years ago.

Index Terms—Deep nets, Learning theory, Spatial sparseness, Massive data

I. INTRODUCTION

With the rapid development of data mining and knowledge discovery, data of massive size are collected in various disciplines [54], including medical diagnosis, financial market analysis, computer vision, natural language processing, time series forecasting, and search engines. These massive data bring additional opportunities to discover subtle data features which cannot be reflected by data of small size while creating a crucial challenge on machine learning to develop learning schemes to realize benefits by exploring the use of massive data. Although numerous learning schemes such as distributed learning [27], localized learning [33] and sub-sampling [15] have been proposed to handle massive data, all these schemes focused on the tractability rather than the benefit of massiveness. Therefore, it remains open to explore the benefits brought from massive data and to develop feasible learning strategies for realizing these benefits.

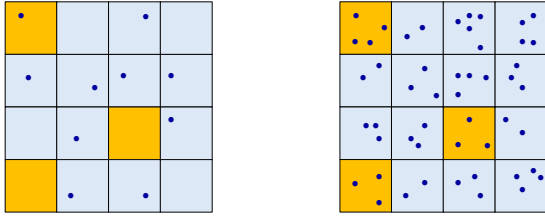
Deep learning [19], characterized by training deep neural networks (deep nets for short) to extract data features by using rich computational resources such as computational power of modern graphical processor units (GPUs) and custom processors, has achieved remarkable success in computer vision [24], speech recognition [25] and game theory [44], practically showing its power in tackling massive data. Recent developments on deep learning theory also provide several

exciting theoretical results to explain the efficiency and rationality of deep learning. In particular, numerous data features such as manifold structures of the input space [42], piecewise smoothness [40], rotation-invariance [6] and sparseness in the frequency domain [41] were proved to be realizable by deep nets but cannot be extracted by shallow neural networks (shallow nets for short) with same order of free parameters. All these interesting studies theoretically verify the necessity of depth in deep learning. The problem is, however, they do not provide any explanations on why deep learning works so well in the era of big data. As is well known, advantages of deep nets over shallow nets were discovered [3] about thirty years ago, implying that studying the approximation capability of deep nets only is not enough to demonstrate the excellent performance of deep learning.

Our purposes in this paper are two folds. On one hand, we aim at showing the necessity of massive data in realizing some data features such as spatial sparseness. As exhibited in Figure 1, if a function is supported on the orange range, then small data content as shown in Figure 1 (a) cannot capture the sparseness of the support. It requires at least one sample point in each sub-cube to reflect the spatial sparseness as shown in Figure 1(b). Such a spatially sparse assumption abounds in numerous application domains such as computer vision [45], signal processing [12] and pattern recognition [20], and several special deep nets have been designed to extract spatially sparse features of data [14]. On the other hand, we study the power of depth of neural networks in realizing the spatial sparseness with the help of massive data. By applying the piecewise linear and continuous property of the rectifier linear unit (ReLU) function, $\sigma(t) := \max\{0, t\}$, we construct a deep net with two hidden layers and finitely many neurons to provide a localized approximation, which is beyond the capability of shallow nets [3], [39], [5]. The localized approximation of deep nets highlights their power in capturing the position information of data inputs. A direct consequence is that deep nets can reflect the spatial sparseness of a function [30]. This property, together with the recently developed approaches in approximating smooth function by deep nets [48], [40], [18], give rise to the feasibility of adopting deep nets to extracting smoothness and spatial sparseness simultaneously.

In particular, we succeed in deriving a quantitative requirement of the data size to extract the spatial sparseness via showing the existence of some learning scheme that can reflect both the smoothness and spatial sparseness, provided the data size achieves certain level. This finding coincides with the well-known sampling theorem in compressed sensing [11]. We then reformulate our sampling theorem in the framework of

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(a) Limitations for small data (b) Advantages for massive data

Fig. 1. The role of data size in realizing spatial sparsity

learning theory [9] by highlighting the important role of data size in deriving optimal learning rates for learning smooth and spatially sparse functions. The established sampling theorem in learning theory theoretically verifies the necessity of massive data in sparseness-related applications and shows that massive data can extract some data features that cannot be reflected by data of small size. Furthermore, we derive almost optimal learning rates for implementing empirical risk minimization (ERM) on deep nets and prove that up to a logarithmic factor, the derived learning rates coincide with those of the sampling theorem. In a nutshell, our results rigorously verify the benefits of massiveness of data in learning smooth and spatially sparse functions, and that deep learning is capable of embodying advantages of massive data.

The rest of this paper is organized as follows. In Section II, we introduce the definition of spatially sparse functions and present the sampling theorem for learning smooth and spatially sparse functions. In Section III, we provide the advantage of deep nets in embodying the benefits of massive data via showing the optimal learning rates for ERM on deep nets. In Section IV, we establish upper bounds of the sampling theorem and learning rates for ERM on deep nets. In Section V, we present the proofs for the lower bounds.

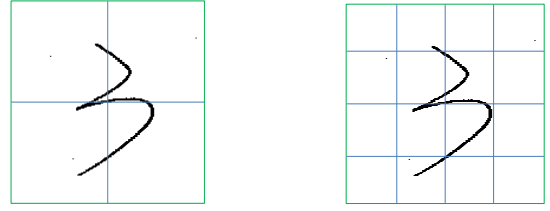
II. SAMPLING THEOREM FOR REALIZING SPATIALLY SPARSE AND SMOOTH FEATURES

In this section, we discuss the benefits of massive data via presenting a sampling theorem in the framework of learning theory.

A. Spatially sparse and smooth functions

Spatial sparseness is a popular data feature which abounds in numerous applications such as handwritten digit recognition [8], magnetic resonance imaging (MRI) analysis [1], image classification [47] and environmental data processing [10]. Different from other sparseness measurements such as the sparseness in the frequency domain [26], [41] and the manifold sparseness [5], spatial sparseness depends heavily on partitions of the input space. Considering handwritten digit recognition as an example, Figure 2 (a) shows that the handwritten digit is not sparse if the partition level is 4. However, if the partition level achieves 16 as shown in Figure 2 (b), the handwritten digit is sparse.

Based on this observation, we present the following definition of spatially sparse functions (see, [30]). Let $\mathbb{I}^d := [0, 1]^d$



(a) Non-sparseness for partitions (b) Sparseness for partitions

Fig. 2. The role of partition in reflecting the spatial sparsity

and $N \in \mathbb{N}$. Partition \mathbb{I}^d by N^d sub-cubes $\{A_j\}_{j=1}^{N^d}$ of side length N^{-1} and with centers $\{\zeta_j\}_{j=1}^{N^d}$. For $s \in \mathbb{N}$ with $s \leq N^d$,

$$\Lambda_s := \{j_\ell : j_\ell \in \{1, 2, \dots, N^d\}, 1 \leq \ell \leq s\},$$

and consider a function f defined on \mathbb{I}^d , if the support of f is contained in $S := \cup_{j \in \Lambda_s} A_j$ for a subset Λ_s of $\{1, 2, \dots, N^d\}$ of cardinality at most s . We then say that f is s -sparse in N^d partitions. In what follows, we take Λ_s to be the smallest subset that satisfies this condition.

Besides the spatial sparseness, we also introduce the smooth property of f , which is a widely used a-priori assumption [48], [40], [29], [5], [51]. Let $c_0 > 0$ and $r = u + v$ with $u \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $0 < v \leq 1$. We say that a function $f : \mathbb{I}^d \rightarrow \mathbb{R}$ is (r, c_0) -smooth if f is u -times differentiable and for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ with $\alpha_1 + \dots + \alpha_d = u$ and $x, x' \in \mathbb{I}^d$, its partial derivative, denoted by

$$f_\alpha^{(u)}(x) = \frac{\partial^u f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(x),$$

satisfies the Lipschitz condition

$$\left| f_\alpha^{(u)}(x) - f_\alpha^{(u)}(x') \right| \leq c_0 \|x - x'\|^v, \quad (1)$$

where $\|x\|$ denotes the Euclidean norm of x . Denote by $Lip^{(r, c_0)}$ the family of (r, c_0) -smooth functions satisfying (1) and by $Lip^{(N, s, r, c_0)}$ the set of all $f \in Lip^{(r, c_0)}$ which are s -sparse in N^d partitions.

B. Sampling theorem for realizing spatially sparse and smooth features

We conduct the analysis in a standard least-square regression framework [9], in which samples $D = \{(x_i, y_i)\}_{i=1}^m$ are drawn independently according to an unknown Borel probability measure ρ on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with $\mathcal{X} = \mathbb{I}^d$ and $\mathcal{Y} \subseteq [-M, M]$ for some $M > 0$. The objective is the regression function defined by

$$f_\rho(x) = \int_{\mathcal{Y}} y d\rho(y|x), \quad x \in \mathcal{X},$$

which minimizes the generalization error

$$\mathcal{E}(f) := \int_{\mathcal{Z}} (f(x) - y)^2 d\rho,$$

where $\rho(y|x)$ denotes the conditional distribution at x induced by ρ . Let ρ_X be the marginal distribution of ρ on \mathcal{X} and $(L_{\rho_X}^2, \|\cdot\|_\rho)$ denote the Hilbert space of ρ_X square-integrable

functions on \mathcal{X} . Then for $f \in L^2_{\rho_X}$, it follows, in view of [9], that

$$\mathcal{E}(f) - \mathcal{E}(f_\rho) = \|f - f_\rho\|_\rho^2. \quad (2)$$

If f_ρ is supported on S but ρ_X is supported on $\mathbb{I}^d \setminus S$, it is impossible to derive a satisfactory learning rate, implying the necessity of restrictions on ρ_X . In this section, we assume ρ_X is the uniform distribution for the sake of brevity. Our result also holds under the classical distortion assumption on ρ_X [50]. Denote by $\mathcal{M}(N, s, r, c_0)$ the set of all distributions satisfying that ρ_X is the uniform distribution and $f_\rho \in Lip^{(N, s, r, c_0)}$. We enter into a competition over Ψ_D , **which denotes the class of all functions derived from the data set D** and define

$$e(N, s, r, c_0) := \sup_{\rho \in \mathcal{M}(N, s, r, c_0)} \inf_{f_D \in \Psi_D} \mathbf{E}(\|f_\rho - f_D\|_\rho^2).$$

The sampling theorem we establish for learning is our first main result.

Theorem 1. *Let $r, c_0 > 0$, $d, s, N, m \in \mathbb{N}$ with $s \leq N^d$. If*

$$\frac{m}{\log m} \geq C^* \frac{N^{2r+2d}}{s}, \quad (3)$$

then

$$\begin{aligned} C_1 m^{-\frac{2r}{2r+d}} \left(\frac{s}{N^d}\right)^{\frac{d}{2r+d}} &\leq e(N, s, r, c_0) \\ &\leq C_2 \left(\frac{m}{\log m}\right)^{-\frac{2r}{2r+d}} \left(\frac{s}{N^d}\right)^{\frac{d}{2r+d}}, \end{aligned} \quad (4)$$

where C^*, C_1, C_2 are constants independent of m, s or N .

The proof of Theorem 1 will be given in Sec. V. The sampling theorem [43] originally focused on deriving the minimal sampling rate that permits a discrete sequence of samples to capture all the information from a continuous-time signal of finite bandwidth in sampling processes. Recent developments [49] imitated the sampling theorem in terms of deriving minimal sizes of samples to represent a signal via some transformations such as wavelet, Fourier and Legendre transformations. The sampling theorem presented in this paper aims at deriving minimal sizes of samples that can achieve the optimal learning rates for some specified learning task. Theorem 1 shows that optimal learning rates for learning spatially sparse and smooth functions are achievable provided (3) holds. The size of samples, as governed in (3), depends on the sparsity level s and partitions numbers N , and increases with respect to N , showing that more partitions require more samples. This coincides with the intuitive observation as shown in Figure 1. Different from the classical results in signal processing [49], the size of samples in (3) decreases with s . This is not surprising, since the established optimal learning rates in (4) increase with s . In other words, the size of samples in our result is to recognize the support of the regression function and thus increases with N while the sparsity s is reflected by optimal learning rates in (4).

The almost optimal learning rate in (4) can be regarded as a combination of two components $m^{-\frac{2r}{2r+d}}$ for the smoothness and $\left(\frac{s}{N^d}\right)^{\frac{d}{2r+d}}$ for the sparseness. If $s = N^d$, meaning that f_ρ is not spatially sparse, then the learning rate derived in

Theorem 1 coincides with the optimal learning rate in learning smooth functions ([17, Chap. 3]), up to a logarithmic factor. If r is extremely small, the learning rate derived in Theorem 1, near to $\frac{s}{N^d}$ due to the uniform assumption on ρ_X , is also the optimal learning rates for learning spatially sparse functions. If m is relatively small with respect to N , i.e. (3) does not hold, then while the smoothness part $m^{-\frac{2r}{2r+d}}$ can be maintained, the sparseness property cannot be captured. This shows the benefit of massive data in learning spatially sparse functions. It should be noted that there is an additional logarithmic term in (4). We believe that it is removable by using different tools from this paper and will consider it as a future work.

III. DEEP NETS IN REALIZING SPATIAL SPARSENESS

In this section, we verify the power of depth for ReLU nets in localized approximation and spatially sparse approximation, and then show that deep nets are able to embody the benefits of massive data in learning spatially sparse and smooth functions.

A. Deep ReLU nets

One of the main reasons for the great success of deep learning is the implementation in terms of deep nets. In comparison with the classical shallow nets, deep nets are significantly better in providing localized approximation [3], manifold learning [42], [5], realizing rotation invariance priors [31], [6], embodying sparsity in the frequency domain [26], [41] and in the spatial domain [30], approximating piecewise smooth functions [40] and capturing the hierarchical structures [36], [23]. However, all these interesting results are not yet sufficient to explain why deep nets perform well in the era of big data.

Let $\sigma(t) := \max\{t, 0\}$ be the rectifier liner unit (ReLU). Deep ReLU nets, i.e. deep nets with the ReLU activation function, is most popular in the current research of deep learning. Due to the non-differentiable property of ReLU, it seems difficult for ReLU nets to approximate smooth functions at the first glance. However, it was shown in [48], [40], [52], [18] that increasing the depth of ReLU nets succeeds in overcoming this problem, **which** provides theoretical foundations in understanding deep ReLU nets.

Denote $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{I}^d$. Let $L \in \mathbb{N}$ and $d_0, d_1, \dots, d_L \in \mathbb{N}$ with $d_0 = d$. For $\vec{h} = (h^{(1)}, \dots, h^{(d_k)})^T \in \mathbb{R}^{d_k}$, define $\vec{\sigma}(\vec{h}) = (\sigma(h^{(1)}), \dots, \sigma(h^{(d_k)}))^T$. Deep ReLU nets with depth L and width d_j in the j -th hidden layer can be mathematically represented as

$$h_{\{d_0, \dots, d_L, \sigma\}}(x) = \vec{a} \cdot \vec{h}_L(x), \quad (5)$$

where

$$\vec{h}_k(x) = \vec{\sigma}(W_k \cdot \vec{h}_{k-1}(x) + \vec{b}_k), \quad k = 1, 2, \dots, L, \quad (6)$$

$\vec{h}_0(x) = x$, $\vec{a} \in \mathbb{R}^{d_L}$, $\vec{b}_k \in \mathbb{R}^{d_k}$, and $W_k = (W_k^{i,j})_{i=1, j=1}^{d_k, d_{k-1}}$ is a $d_k \times d_{k-1}$ matrix. Denote by $\mathcal{H}_{\{d_0, \dots, d_L, \sigma\}}$ the set of all these deep ReLU nets. The structures of deep nets are reflected by the weight matrices W_k and threshold vectors \vec{b}_k , $k = 1, \dots, L$. For example, taking the special form of Toeplitz-type weight matrices leads to the deep convolutional nets [51], [52], [53], full matrices correspond to deep fully

connected nets [13], and tree-type sparse matrices imply deep nets with tree structures [6], [7]. In this paper, we do not focus on the structure selection of deep nets, but rather on the existence of some deep net structure for realization of the sampling theorem established in Theorem 1.

B. Deep ReLU nets for localized approximation

Localized approximation is an important property of neural networks and is a crucial stepping stone in approximating piecewise smooth functions [40] and spatially sparse functions [30]. The localized approximation of a neural network allows the target function to be modified in any small region of the Euclidean space by adjusting a few neurons, rather than the entire network. It was originally proposed in [3, Def. 2.1] to demonstrate the power of depth for deep nets with sigmoid-type activation functions. The main conclusion in [3] is that deep nets only with two hidden layers and $2d + 1$ neurons can provide localized approximation, while shallow nets fail for $d \geq 2$, even for the simplest Heaviside activation function. In this section, we prove that deep ReLU nets with two hidden layers and $4d + 1$ neurons are capable of providing localized approximation.

For $a, b \in \mathbb{R}$ with $a < b$, define a trapezoid-shaped function $T_{\tau, a, b}$ with a parameter $0 < \tau \leq 1$ as

$$T_{\tau, a, b}(t) := \frac{1}{\tau} \left\{ \begin{aligned} &\sigma(t - a + \tau) - \sigma(t - a) \\ &- \sigma(t - b) + \sigma(t - b - \tau) \end{aligned} \right\}. \quad (7)$$

Then the definition of σ yields

$$T_{\tau, a, b}(t) = \begin{cases} 1, & \text{if } a \leq t \leq b, \\ 0, & \text{if } t \geq b + \tau, \text{ or } t \leq a - \tau, \\ \frac{b + \tau - t}{\tau}, & \text{if } b < t < b + \tau, \\ \frac{t - a + \tau}{\tau}, & \text{if } a - \tau < t < a. \end{cases} \quad (8)$$

We may then consider

$$\mathcal{L}_{a, b, \tau}(x) := \sigma \left(\sum_{j=1}^d T_{\tau, a, b}(x^{(j)}) - (d - 1) \right). \quad (9)$$

The following proposition presents the localized approximation property of $\mathcal{N}_{a, b, \tau}$.

Proposition 1. *Let $a < b$, $0 < \tau \leq 1$ and $\mathcal{L}_{a, b, \tau}$ be defined by (9). Then we have $0 \leq \mathcal{L}_{a, b, \tau}(x) \leq 1$ for all $x \in \mathbb{I}^d$ and*

$$\mathcal{L}_{a, b, \tau}(x) = \begin{cases} 0, & \text{if } x \notin [a - \tau, b + \tau]^d, \\ 1, & \text{if } x \in [a, b]^d. \end{cases} \quad (10)$$

The proof of Proposition 1 is postponed to Section IV. Similar approximation results for deep nets with sigmoid-type activation functions and $2d + 1$ neurons have been established in [3], [42], [30]. The representation in Proposition 1 is better because the expression for $x \in [a, b]^d$ and $x \notin [a - \tau, b + \tau]^d$ is exact. For arbitrary $N^* \in \mathbb{N}$, partition \mathbb{I}^d into $(N^*)^d$ sub-cubes $\{B_k\}_{k=1}^{(N^*)^d}$ of side length $1/N^*$ and with centers $\{\xi_k\}_{k=1}^{(N^*)^d}$. Write $B_{k, \tau} := [\xi_k + [-1/(2N^*) - \tau, 1/(2N^*) + \tau]^d] \cap \mathbb{I}^d$. It is obvious that $B_k \subset B_{k, \tau}$. Define $\mathcal{N}_{1, N^*, \xi, \tau} : \mathbb{I}^d \rightarrow \mathbb{R}$ for $\xi \in \mathbb{I}^d$ by

$$\mathcal{N}_{1, N^*, \xi, \tau}(x) = \mathcal{L}_{-1/(2N^*), 1/(2N^*), \tau}(x - \xi). \quad (11)$$

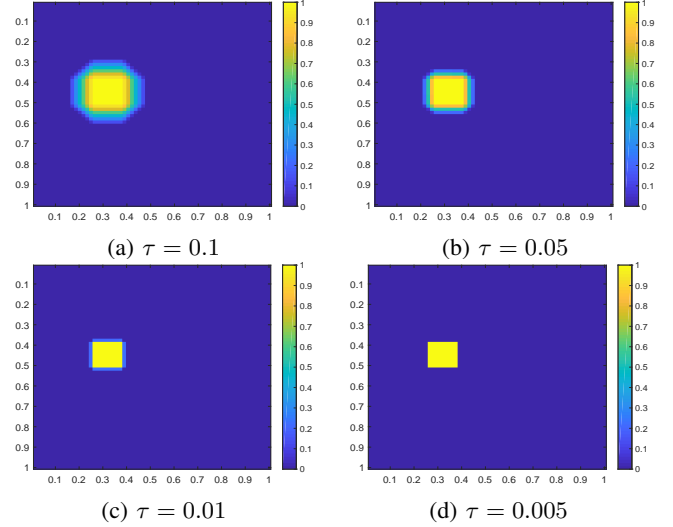


Fig. 3. The localized approximation based on a cubic partition of $[0, 1]^2$ with side length $1/8$ for the deep net constructed in (11) with $\xi = (3/16, 5/16)$

In view of Proposition 1, (8) and (11), we have $|\mathcal{N}_{1, N^*, \xi_k, \tau}(x)| \leq 1$ for all $x \in \mathbb{I}^d$, $k \in \{1, \dots, (N^*)^d\}$ and

$$\mathcal{N}_{1, N^*, \xi_k, \tau}(x) = \begin{cases} 0, & \text{if } x \notin \tilde{B}_{k, \tau}, \\ 1, & \text{if } x \in B_k. \end{cases} \quad (12)$$

As shown in Figure 3, the parameter τ determines the size of $\tilde{B}_{k, \tau}$, and thus affects the performance of localized approximation for the constructed deep nets in (11). However, it does not mean the smaller τ the better, since the norms of weights decrease with respect to τ , which may result in extremely large capacity of deep ReLU nets for too small τ .

C. Deep ReLU nets for spatially sparse approximation

The localized approximation established in Proposition 1 demonstrates the power of deep ReLU nets with two hidden layers to **reflect** some spatial information of the input. A direct consequence is that deep ReLU nets succeed in **realizing** the spatially sparse property of functions and also maintaining the capability of deep ReLU nets in approximating smooth functions. On one hand, spatial sparseness defined in this paper is built upon a cubic partition of \mathbb{I}^d , i.e. $\mathbb{I}^d = \cup_{j=1}^{N^d} A_j$. If $N^* \geq N$, then $A_j \subseteq \cup_{k: A_j \cap B_k \neq \emptyset} B_k$ can be recognized by the localized approximation of $\mathcal{N}_{1, N^*, \xi_k, \tau}$. Figure 4 demonstrates that for small enough τ , summations of $\mathcal{N}_{1, N^*, \xi_k, \tau}$ with different k can reflect the spatial sparseness for $N^* = N$. On the other hand, due to the localized approximation of $\mathcal{N}_{1, N^*, \xi_k, \tau}(x)$, for any $x \in \mathbb{I}^d$, there is at most 2^d indices k_j with $\mathcal{N}_{1, N^*, \xi_{k_j}, \tau}(x) = 1$ for $j = 1, 2, \dots, 2^d$ and $|\mathcal{N}_{1, N^*, \xi_k, \tau}(x)|$ extremely small for $k \neq k_j$. Then, for large enough N^* , the smoothness of f leads to small approximation error for $\left| f(x) - \sum_{k=1}^{(N^*)^d} f(\xi_k) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \right|$.

With the above observations, we find that deep ReLU nets are capable of realizing both the smoothness and spatial sparseness, which is beyond the capability of shallow ReLU nets [3], [48]. The following proposition is the main result in this subsection.

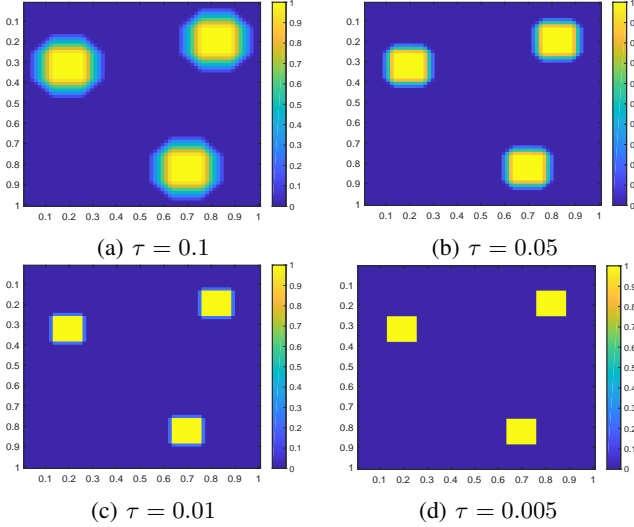


Fig. 4. Realizing spatial sparseness by summations of the deep net constructed in (11) with different k

Proposition 2. *Let $1 \leq p < \infty$, $r, c_0 > 0$, $N, s, d \in \mathbb{N}$ with $s \leq N^d$ and $N^* \geq \max\{4N, \tilde{C}\}$. Then there exists a deep ReLU net structure with $\lceil 25 + 4r/d + 2r^2/d + 10r \rceil$ inner layers and at most $C_1^*(N^*)^d$ free parameters, such that for any $f \in \text{Lip}^{(N, s, r, c_0)}$ and any $0 < \tau \leq \frac{s}{2N^d(N^*)^{1+pr}}$, there is a deep ReLU net $\mathcal{N}_{3, N^*, \tau}$ with the aforementioned structure and free parameters bounded by*

$$\tilde{B}^* := C_3 \max\{1/\tau, (N^*)^{(2d+r)\gamma}\}. \quad (13)$$

such that

$$\|f - \mathcal{N}_{3, N^*, \tau}\|_{L^p(\mathbb{I}^d)} \leq C_4 (N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p}, \quad (14)$$

and

$$\|\mathcal{N}_{3, N^*, \tau}\|_{L^\infty(\mathbb{I}^d)} \leq C_5, \quad (15)$$

where $\gamma, \tilde{C}, C_1^*, C_3, C_4, C_5$ are constants depending only on c_0, r, d and $\|f\|_{L^\infty(\mathbb{I}^d)}$.

The proof of Proposition 2 will be given in Section IV. Approximating functions in $\text{Lip}^{(r, c_0)}$ is a classical topic in neural network approximation. It is shown in [34] that for shallow nets with C^∞ sigmoid type activation functions and $(N^*)^d$ free parameters, an approximation rate of order $(N^*)^{-r}$ can be achieved. Furthermore, [32], [28] provide a lower bound. Although these nice results show the excellent approximation capability of shallow nets, the weights of shallow nets in [34], [32] are extremely large, resulting in extremely large capacity. With such extremely large weights, it follows from the results in [32], [21] that there exists a deep net with two hidden layers and finitely many neurons possessing the universal approximation property. The problem of extremely large weights can be avoided by deepening the neural networks. In fact, it can be found in [48], [40], [18] that similar results hold for deep ReLU nets with a few hidden layers and controllable weights, i.e., weights increasing polynomially fast with respect to the number of free parameters. Our Proposition 2 also implies this finding by setting $s = (N^*)^d$ and larger value

of τ . It will be shown in the next subsection that controllable weights play a crucial role in deriving small variance as well as fast learning rates for implementing ERM on deep ReLU nets.

The approximation rates established in (14) not only reveal the power of depth in approximating smooth functions, but also exhibit the advantage of deep ReLU nets in embodying the spatial sparseness by means of multiplying an additional $\left(\frac{s}{N^d}\right)^{1/p}$ on the optimal approximation rates $(N^*)^{-r}$ for smooth functions. Noting that shallow nets with the Heaviside activation function [3] cannot provide localized approximation, corresponding to a special case of $s = 1$, Proposition 2 shows the power of depth of deep ReLU net in spatially sparse approximation.

D. Realizing optimal learning rates in sampling theorem by deep nets

In this subsection, we aim at developing a learning scheme to take advantage of the power of deep ReLU nets in realizing the spatial sparseness and smoothness. Denote by $\mathcal{H}_{n, L}$ the collection of deep nets that possess the structure in Proposition 2 with

$$L = \lceil 25 + 4r/d + 2r^2/d + 10r \rceil, \quad \text{and} \quad n = C_1^*(N^*)^d. \quad (16)$$

Define

$$\mathcal{H}_{n, L, \mathcal{R}} := \{h_{n, L} \in \mathcal{H}_{n, L} : |w_k^{i, j}|, |b_k^i|, |a_i| \leq \mathcal{R}, \\ 1 \leq i \leq d_k, 1 \leq j \leq d_{k-1}, 1 \leq k \leq L\}, \quad (17)$$

where

$$\mathcal{R} := C_3 \max \left\{ \frac{2N^d(N^*)^{1+pr}}{s}, (N^*)^{2\gamma d + \gamma r} \left(\frac{N^d}{s}\right)^{\gamma/p} \right\}. \quad (18)$$

Then, it turns out that

$$\mathcal{N}_{3, N^*, \tau} \in \mathcal{H}_{n, L, \mathcal{R}} \quad (19)$$

with $\tau = \frac{s}{2N^d(N^*)^{1+pr}}$.

We consider the generalization error estimates for implementing ERM on $\mathcal{H}_{n, L, \mathcal{R}}$ as follows:

$$f_{D, n, L} := \arg \min_{f \in \mathcal{H}_{n, L, \mathcal{R}}} \frac{1}{m} \sum_{i=1}^m [f(x_i) - y_i]^2. \quad (20)$$

Since $|y_i| \leq M$, it is natural to project the final output $f_{D, n, L}$ to the interval $[-M, M]$ by the truncation operator $\pi_M f_{D, n, L}(x) := \text{sign}(f_{D, n, L}(x)) \min\{|f_{D, n, L}(x)|, M\}$.

Let $p \geq 2$ and J_p be the identity mapping

$$L^p(\mathcal{X}) \xrightarrow{J_p} L_{\rho_X}^2.$$

and $D_{\rho_X, p} = \|J_p\|$. Then $D_{\rho_X, p}$ is called the distortion of ρ_X (with respect to the Lebesgue measure) [50], which measures how much ρ_X distorts the Lebesgue measure. In our analysis, we assume $D_{\rho_X, p} < \infty$, which holds for the uniform distribution for all $p \geq 2$ obviously. According to the definition, for each $f \in L_{\rho_X}^2 \cap L^p(\mathbb{I}^d)$, we have

$$\|f\|_\rho \leq D_{\rho_X, p} \|f\|_{L^p(\mathcal{X})}. \quad (21)$$

The following theorem with proof to be given in Section V shows that the simple ERM strategy (20) based on deep ReLU nets has the capability of realizing the optimal learning rates established in Theorem 1.

Theorem 2. *Let $f_{D,n,L}$ be defined by (20) with L, n satisfying (16) and \mathcal{R} satisfying (18). Suppose that*

$$(N^*)^{2r+d} \sim m \left(\frac{s}{Nd} \right)^{\frac{2}{p}} / \log m, \quad (22)$$

and

$$\frac{m}{\log m} \geq C^* \frac{N^{\frac{2d+2rp+dp}{(2r+d)p}}}{s^{\frac{2}{2rp+dp}}}. \quad (23)$$

Then

$$\begin{aligned} & C_1 m^{-\frac{2r}{2r+d}} \left(\frac{s}{Nd} \right)^{\frac{d}{2r+d}} \\ & \leq \sup_{f_\rho \in \text{Lip}(N,s,r,c_0)} \mathbf{E} \{ \mathcal{E}(\pi_M f_{D,n,L^*}) - \mathcal{E}(f_\rho) \} \\ & \leq C_6 \left(\frac{m}{\log m} \right)^{\frac{-2r}{2r+d}} \left(\frac{s}{Nd} \right)^{\frac{2}{p} - \frac{2r}{2r+d}}, \end{aligned} \quad (24)$$

where C_1, C_6 are constants independent of m, s , or N and $a \sim b$ with $a, b \geq 0$ denotes that there exist positive absolute constants \hat{C}_1, \hat{C}_2 such that $\hat{C}_1 a \leq b \leq \hat{C}_2 a$.

If ρ_X is the uniform distribution, then (21) holds with $p = 2$ and $D_{\rho_X, p} = 1$. Hence, if $f_\rho \in \text{Lip}(N, s, r, c_0)$, we have

$$\begin{aligned} & \mathbf{E} \{ \mathcal{E}(\pi_M f_{D,n,L^*}) - \mathcal{E}(f_\rho) \} \\ & \leq C_6 m^{-\frac{2r}{2r+d}} (\log m)^{2r/(r+d)} \left(\frac{s}{Nd} \right)^{\frac{d}{2r+d}}, \end{aligned} \quad (25)$$

which coincides with the optimal learning rates in Theorem 1 up to a logarithmic factor. Theorem 2 thus presents a theoretical verification on the success of deep learning in spatial sparseness related applications for massive data. In particular, it presents an intuitive explanation on why deep learning performs so well in handwritten digit recognition [2]. As shown in Figure 2, high-resolution of a figure implies large size of data, which admits an extremely large partitions for the input space with small sparsity s . Then, the additional term $\left(\frac{s}{Nd} \right)^{\frac{d}{2r+d}}$ in Theorem 2 yields a small generalization error.

Learning spatially sparse and smooth functions was originally studied in [30] and similar learning rate as that in Theorem 2 has been derived. In comparison with [30], there are three novelties of our work. The first is that we deduce lower bounds for learning these functions and show the optimality for the derived learning rates, while [30] only focused on upper bounds. The second is that the range of r in our study is $r > 0$, while that in [30] is $0 < r \leq 1$. In view of the discussion in [48], any twice-differentiable nonlinear function defined on \mathbb{I}^d cannot be ε -approximated by ReLU networks of fixed depth L with the number of free parameters less than $c\varepsilon^{-1/(2(L-2))}$ with a positive constant c depending only on d . The depth is thus necessary for extending the range from $0 < r \leq 1$ to $r > 0$. Therefore, more layers are required in our analysis to show the advantage of deep nets. Finally, we would like to point out that the activation function in the present paper is the widely used ReLU function, while the

activation functions in [30] are hybrid functions including the Heaviside function in the first layer and continuous sigmoid-type function in other layers. We end this section by two important remarks.

Remark 1. *Besides the deep ReLU nets studied in the present paper, other approaches including the radial basis function networks [38], partition estimates [17, Chap.4], Nadaraya-Watson estimates [17, Chap.5] and many others [35], [37] may perform well in approximating and learning spatially sparse and smooth functions. We highlight that our purpose is not to show the uniqueness of deep nets, but rather to demonstrate the power of depth in comparison with the classical shallow nets. Our results, together with the previous studies [42], [48], [40], [41], [18] show the versatility of deep ReLU nets in terms of their outperformance in approximating and learning numerous functions.*

Remark 2. *Since ReLU is the most popular activation function in deep learning [13], we focus on the outperformance of deep ReLU nets in the present paper. We believe that our results also hold for deep nets with other activation functions such as the sigmoid functions in [3], squared ReLU in [6] and hybrid functions in [30].*

IV. UPPER BOUND ESTIMATES

This section is devoted to the proof of Proposition 1, Proposition 2, and the upper bounds in (24) and (4). It should be noted that the upper bound in (4) is a direct corollary of that of (24), with $p = 2$.

A. Proofs of Proposition 1

Proof of Proposition 1: For $x \in \mathbb{I}^d$, it follows from (8) that $0 \leq T_{\tau,a,b}(x^{(j)}) \leq 1$ for any $j \in \{1, \dots, d\}$. This implies that $\sum_{j=1}^d T_{\tau,a,b}(x^{(j)}) \leq d$ and consequently $0 \leq \mathcal{L}_{a,b,\tau}(x) \leq 1$. If $x \notin [a-\tau, b+\tau]^d$, there is at least one $j_0 \in \{1, \dots, d\}$ such that $x^{(j_0)} \notin [a-\tau, b+\tau]$. This together with (8) shows that $T_{\tau,a,b}(x^{(j_0)}) = 0$. Therefore $\sum_{j=1}^d T_{\tau,a,b}(x^{(j)}) \leq d-1$, which implies $\mathcal{L}_{a,b,\tau}(x) = 0$. If $x \in [a, b]^d$, then $x^{(j)} \in [a, b]$ for every $j \in \{1, \dots, d\}$. Hence, it follows from (8) that $T_{\tau,a,b}(x^{(j)}) = 1$ for every $j \in \{1, \dots, d\}$, which implies that $\sum_{j=1}^d T_{\tau,a,b}(x^{(j)}) = d$ and $\mathcal{L}_{a,b,\tau}(x) = 1$. This completes the proof of Proposition 1. ■

B. Proof of Proposition 2

Before presenting the proof of Proposition 2, we need several lemmas. The first one can be found in [22, Lemma 1].

Lemma 1. *Let $r = u + v$ with $u \in \mathbb{N}_0$ and $0 < v \leq 1$. If $f \in \text{Lip}^{(r,c_0)}$, $x_0 \in \mathbb{R}^d$ and $p_{u,x_0,f}$ is the Taylor polynomial of f with degree u at x_0 , i.e.,*

$$\begin{aligned} p_{u,x_0,f}(x) &= \sum_{k_1+\dots+k_d \leq u} \frac{1}{k_1! \dots k_d!} \partial^{k_1} x^{(1)} \dots \partial^{k_d} x^{(d)} f(x_0) \\ & \quad (x^{(1)} - x_0^{(1)})^{k_1} \dots (x^{(d)} - x_0^{(d)})^{k_d}, \end{aligned} \quad (26)$$

then

$$|f(x) - p_{u, x_0, f}(x)| \leq c_1 \|x - x_0\|^r, \quad \forall x \in \mathbb{I}^d, \quad (27)$$

where c_1 is a constant depending only on r , c_0 and d .

For $\tau > 0$, define the localized Taylor polynomials by

$$\mathcal{N}_{2, N^*, \tau}(x) := \sum_{k=1}^{(N^*)^d} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x), \quad (28)$$

where $\mathcal{N}_{1, N^*, \xi_k, \tau}$ is given in (11). In the following lemma, we present an upper bound estimate for approximating functions in $Lip^{(N, s, r, c_0)}$ by $\mathcal{N}_{2, N^*, \tau}$.

Lemma 2. *Let $1 \leq p < \infty$ and $N^* \geq 4N$. If $f \in Lip^{(N, s, r, c_0)}$ with $N, s \in \mathbb{N}$, $r > 0$ and $c_0 > 0$, then for any $0 < \tau \leq \frac{s}{2N^d(N^*)^{1+pr}}$, we have*

$$\|f - \mathcal{N}_{2, N^*, \tau}\|_{L^p(\mathbb{I}^d)} \leq c_2 (N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p} \quad (29)$$

and

$$\|\mathcal{N}_{2, N^*, \tau}\|_{L^\infty(\mathbb{I}^d)} \leq c_3, \quad (30)$$

where c_2, c_3 are constants dependent only on d , r , c_0 and $\|f\|_{L^\infty(\mathbb{I}^d)}$.

Proof: Observe that $\mathbb{I}^d = \bigcup_{k=1}^{(N^*)^d} B_k$. Then for each $x \in \mathbb{I}^d$, let k_x be the smallest k such that $x \in B_{k_x}$. Note that k_x is unique (the last restriction is for those points x on boundaries of cubes B_k). It follows from (28) that

$$\begin{aligned} & f(x) - \mathcal{N}_{2, N^*, \tau}(x) \\ &= f(x) - \sum_{k=1}^{(N^*)^d} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \\ &= f(x) - p_{u, \xi_{k_x}, f}(x) - \sum_{k \neq k_x} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \\ &+ p_{u, \xi_{k_x}, f}(x) [1 - \mathcal{N}_{1, N^*, \xi_{k_x}, \tau}(x)]. \end{aligned}$$

But (12) implies that $1 - \mathcal{N}_{1, N^*, \xi_{k_x}, \tau}(x) = 0$. Thus,

$$\begin{aligned} & \|f - \mathcal{N}_{2, N^*, \tau}\|_{L^p(\mathbb{I}^d)} \leq \|f - p_{u, \xi_{k_x}, f}\|_{L^p(\mathbb{I}^d)} \\ &+ \left\| \sum_{k \neq k_x} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \right\|_{L^p(\mathbb{I}^d)}. \quad (31) \end{aligned}$$

We first estimate the first term on the right-hand side of (31). For $j \in \Lambda_s$, define

$$\tilde{\Lambda}_j := \{k \in \{1, \dots, (N^*)^d\} : B_k \cap A_j \neq \emptyset\}. \quad (32)$$

Since $\{A_j\}_{j=1}^{N^d}$ and $\{B_k\}_{k=1}^{(N^*)^d}$ are cubic partitions of \mathbb{I}^d and $N^* \geq 4N$, we have

$$|\tilde{\Lambda}_j| \leq \left(\frac{N^*}{N} + 2\right)^d \leq \left(\frac{2N^*}{N}\right)^d, \quad \forall j \in \Lambda_s. \quad (33)$$

In view of (32), we obtain

$$\mathbb{I}^d \subseteq \left[\bigcup_{j \in \Lambda_s} \left(\bigcup_{k \in \tilde{\Lambda}_j} B_k \right) \right] \cup \left[\left(\bigcup_{k \in \{1, \dots, (N^*)^d\} \setminus (\bigcup_{j \in \Lambda_s} \tilde{\Lambda}_j)} B_k \right) \right]. \quad (34)$$

Then,

$$\begin{aligned} & \|f - p_{u, \xi_{k_x}, f}\|_{L^p(\mathbb{I}^d)} = \int_{\mathbb{I}^d} |f(x) - p_{u, \xi_{k_x}, f}(x)|^p dx \\ & \leq \left[\sum_{j \in \Lambda_s} \sum_{k \in \tilde{\Lambda}_j} + \sum_{k \in \{1, \dots, (N^*)^d\} \setminus (\bigcup_{j \in \Lambda_s} \tilde{\Lambda}_j)} \right] \\ & \int_{B_k} |f(x) - p_{u, \xi_{k_x}, f}(x)|^p dx. \quad (35) \end{aligned}$$

From (32) again, for any $k \in \{1, \dots, (N^*)^d\} \setminus (\bigcup_{j \in \Lambda_s} \tilde{\Lambda}_j)$, we have $B_k \cap S = \emptyset$, which together with (26) and $f \in Lip^{(N, s, r, c_0)}$ yields $f(x) = p_{u, \xi_{k_x}, f}(x) = 0$ for $x \in B_k$. Hence,

$$\sum_{k \in \{1, \dots, (N^*)^d\} \setminus (\bigcup_{j \in \Lambda_s} \tilde{\Lambda}_j)} \int_{B_k} |f(x) - p_{u, \xi_{k_x}, f}(x)|^p dx = 0. \quad (36)$$

Since $f \in Lip^{(N, s, r, c_0)}$, it follows from Lemma 1 and (33) that

$$\begin{aligned} & \sum_{j \in \Lambda_s} \sum_{k \in \tilde{\Lambda}_j} \int_{B_k} |f(x) - p_{u, \xi_{k_x}, f}(x)|^p dx \\ & \leq c_1^p \sum_{j \in \Lambda_s} \sum_{k \in \tilde{\Lambda}_j} \int_{B_k} \|x - \xi_{k_x}\|^{pr} dx \\ & \leq c_1^p 2^d d^{pr/2} (N^*)^{-pr} \frac{s}{N^d}. \quad (37) \end{aligned}$$

Inserting (37) and (36) into (35), we obtain

$$\|f - p_{u, \xi_{k_x}, f}\|_{L^p(\mathbb{I}^d)} \leq c_1 2^{d/p} d^{r/2} (N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p}. \quad (38)$$

Now we estimate the second term of the right-hand side of (31). For each $k' \in \{1, \dots, (N^*)^d\}$, define

$$\Xi_{k'} := \{k \in \{1, \dots, (N^*)^d\} : \tilde{B}_{k, \tau} \cap B_{k'} \neq \emptyset, k \neq k'\}. \quad (39)$$

Since $0 < \tau \leq \frac{1}{2N^*}$, it is easy to verify that

$$|\Xi_{k'}| \leq 3^d - 1, \quad \forall k' \in \{1, \dots, (N^*)^d\}. \quad (40)$$

Noting further that

$$\begin{aligned} & \left\| \sum_{k \neq k_x} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \right\|_{L^p(\mathbb{I}^d)}^p \\ & \leq \sum_{k'=1}^{(N^*)^d} \int_{B_{k'}} \left| \sum_{k \neq k_x} p_{u, \xi_k, f}(x) \mathcal{N}_{1, N^*, \xi_k, \tau}(x) \right|^p dx, \quad (41) \end{aligned}$$

we obtain, from (39), (40), (12) and $|\mathcal{N}_{1,N^*,\xi_k,\tau}(x)| \leq 1$, that

$$\begin{aligned} & \int_{B_{k'}} \left| \sum_{k \neq k_x} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \\ &= \int_{B_{k'}} \left| \sum_{k \in \Xi_{k'}} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \\ &\leq \max_{1 \leq k \leq (N^*)^d} \|p_{u,\xi_k,f}\|_{L^\infty(\mathbb{I}^d)}^p \\ &\times \sum_{\ell \in \Xi_{k'}} \int_{\tilde{B}_{\ell,\tau} \cap B_{k'}} \left| \sum_{k \in \Xi_{k'}} \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \\ &\leq 3^{dp} \max_{1 \leq k \leq (N^*)^d} \|p_{u,\xi_k,f}\|_{L^\infty(\mathbb{I}^d)}^p \sum_{\ell \in \Xi_{k'}} \int_{\tilde{B}_{\ell,\tau} \cap B_{k'}} dx. \end{aligned}$$

But $k' \notin \Xi_{k'}$ implies that for any $\ell \in \Xi_{k'}$,

$$\int_{\tilde{B}_{\ell,\tau} \cap B_{k'}} dx \leq (1/N^* + 2\tau)^d - (1/N^*)^d \leq 2d\tau(N^*)^{1-d}, \quad (42)$$

where the mean value theorem is applied to yield the last inequality. Thus,

$$\begin{aligned} & \int_{B_{k'}} \left| \sum_{k \neq k_x} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \\ &\leq 2d3^{d(p+1)} \max_{1 \leq k \leq (N^*)^d} \|p_{u,\xi_k,f}\|_{L^\infty(\mathbb{I}^d)}^p \tau(N^*)^{1-d}. \end{aligned}$$

Plugging the above estimate into (41), we conclude from $0 < \tau \leq (N^*)^{-1-pr} \left(\frac{s}{2N^d}\right)$ that

$$\begin{aligned} & \left\| \sum_{k \neq k_x} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right\|_{L^p(\mathbb{I}^d)} \\ &\leq d^{1/p} 3^{2d} \max_{1 \leq k \leq (N^*)^d} \|p_{u,\xi_k,f}\|_{L^\infty(\mathbb{I}^d)} (N^*)^{-r} \left(\frac{s}{N^d}\right)^{\frac{1}{p}}. \end{aligned} \quad (43)$$

Inserting (38) and (43) into (31) and noting that

$$\max_{1 \leq k \leq (N^*)^d} \|p_{u,\xi_k,f}\|_{L^\infty(\mathbb{I}^d)} \leq \|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2},$$

from (27), we deduce that

$$\|f - \mathcal{N}_{2,N^*,\tau}\|_{L^p(\mathbb{I}^d)} \leq c_2 (N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p},$$

with $c_2 := c_1 2^{d/p} d^{r/2} + d^{1/p} 3^{2d} (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2})$. This proves (29).

We now turn to prove (30). First, (28) and (12) imply that for $x \in \mathbb{I}^d$,

$$\begin{aligned} \mathcal{N}_{2,N^*,\tau}(x) &= \sum_{k=1}^{(N^*)^d} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \\ &= \sum_{k: \tilde{B}_{k,\tau} \cap B_{k_x} \neq \emptyset} p_{u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x). \end{aligned}$$

Since $0 < \tau \leq 1/(2N^*)$, it follows from (40) and $0 \leq \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \leq 1$ that

$$|\mathcal{N}_{2,N^*,\tau}(x)| \leq 3^d (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2}) =: c_3, \quad \forall x \in \mathbb{I}^d.$$

This completes the proof of Lemma 2. \blacksquare

The following ‘‘product-gate’’ property for deep ReLU nets can be found in [18].

Lemma 3. *Let $\theta > 0$ and $\tilde{L} \in \mathbb{N}$ with $\tilde{L} > (2\theta)^{-1}$. For any $\ell \in \{2, 3, \dots\}$ and $\varepsilon \in (0, 1)$, there exists a deep ReLU net $\tilde{\times}_\ell$ with $2\ell\tilde{L} + 8\ell$ layers and at most $c_4 \ell^\theta \varepsilon^{-\theta}$ free parameters bounded by $\ell^\gamma \varepsilon^{-\gamma}$, such that*

$$|t_1 t_2 \cdots t_\ell - \tilde{\times}_\ell(t_1, \dots, t_\ell)| \leq \varepsilon, \quad \forall t_1, \dots, t_\ell \in [-1, 1],$$

where c_4 and γ are constants depending only on θ and \tilde{L} .

For $\beta \in \mathbb{N}_0$ and $B > 0$, define

$$\mathcal{P}_{\beta,B}^d := \left\{ \sum_{|\alpha| \leq \beta} c_\alpha x^\alpha : |c_\alpha| \leq B \right\},$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $x^\alpha = (x^{(1)})^{\alpha_1} \cdots (x^{(d)})^{\alpha_d}$. The following lemma was proved in [18].

Lemma 4. *Let $\beta \in \mathbb{N}_0$, $B > 0$, $\theta > 0$ and $\tilde{L} \in \mathbb{N}$ with $\tilde{L} > (2\theta)^{-1}$. For every $P \in \mathcal{P}_{\beta,B}^d$ and $0 < \varepsilon < 1$, there is a deep ReLU net structure with $2\beta\tilde{L} + 8\beta + 1$ layers and at most $\beta^d + c_4(\beta^{d+1}B)^\theta \varepsilon^{-\theta}$ free parameters bounded by $\max\{B, (\beta^{d+1}B)^\gamma \varepsilon^{-\gamma}\}$ such that for any $P \in \mathcal{P}_{\beta,B}^d$ there exists a deep ReLU net h_P with the aforementioned structure that satisfies*

$$|P(x) - h_P(x)| \leq \varepsilon, \quad \forall x \in \mathbb{I}^d.$$

Let c_5 be a constant that satisfies

$$\begin{aligned} & c_4 2^{\frac{d}{2r+d}} c_5^{-\frac{d}{2r+d}} + 4d + 1 + r^d \\ &+ c_4 (u^{d+1} (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2}))^{\frac{d}{r+d}} c_5^{-\frac{d}{r+d}} \\ &\leq 2(4d + 1 + r^d). \end{aligned}$$

For $N^* > \max\{c_5^{1/(d+r)}, 1\}$, let $\tilde{\times}_2$ be the deep net as introduced in Lemma 3 with $\ell = 2$, $\theta = \frac{d}{2d+r}$, $\varepsilon = c_5 (N^*)^{-2d-r}$ and let $\tilde{L} = \lceil 2 + r/d \rceil$. Denote by $h_{p_{u,\xi_k,f}}$ the deep ReLU net in Lemma 4 with $P = p_{u,\xi_k,f}$, $\beta = u$, $B = \|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2}$, $\varepsilon = c_5 (N^*)^{-r-d}$, $\theta = d/(r+d)$, and $\tilde{L} = \lceil 1 + r/d \rceil$. From Lemma 4, we have, for any $x \in \mathbb{I}^d, k = 1, \dots, (N^*)^d$, that

$$|h_{p_{u,\xi_k,f}}(x)| \leq |p_{u,\xi_k,f}(x)| + 1 \leq \|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1. \quad (44)$$

Next consider

$$\begin{aligned} \mathcal{N}_{3,N^*,\tau}(x) &:= (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1) \\ &\times \sum_{k=1}^{(N^*)^d} \tilde{\times}_2 \left(\frac{h_{p_{u,\xi_k,f}}(x)}{\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1}, \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right). \end{aligned} \quad (45)$$

Noting that the parameters of the deep nets $\tilde{\times}_2(t_1, t_2)$ are independent of $t_1, t_2 \in [-1, 1]$, we conclude that $\mathcal{N}_{3,N^*,\tau}$ is a deep net with $\lceil 25 + 4r/d + 2r^2/d + 10r \rceil$ layers and at most $C_1^* (N^*)^d$ free parameters with $C_1^* := 2(4d + 1 + r^d)$ that are bounded by \tilde{B}^* defined by (13) with $C_3 := 2r^{d+1} (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2})$. With these preparations, we can now prove Proposition 2 as follows.

Proof of Proposition 2: By applying the triangle inequality, we have

$$\begin{aligned}
& \|f - \mathcal{N}_{3,N^*,\tau}\|_{L^p(\mathbb{I}^d)} \\
\leq & \left\| \mathcal{N}_{2,N^*,\tau}(x) - \sum_{k=1}^{(N^*)^d} h_{p_u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right\|_{L^p(\mathbb{I}^d)} \\
& + \left\| \sum_{k=1}^{(N^*)^d} h_{p_u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) - \mathcal{N}_{3,N^*,\tau}(x) \right\|_{L^p(\mathbb{I}^d)} \\
& + \|f - \mathcal{N}_{2,N^*,\tau}(x)\|_{L^p(\mathbb{I}^d)} \\
=: & I_1 + I_2 + I_3. \tag{46}
\end{aligned}$$

It follows from $p \geq 1$, $N^* \geq 4N$ and Lemma 3 with $\ell = 2$, $\theta = \frac{d}{2d+r}$, $\varepsilon = c_5(N^*)^{-2d-r}$ and $\tilde{L} = \lceil 2 + r/d \rceil$ that

$$\begin{aligned}
I_2 & \leq c_5(\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1)(N^*)^{-r-d} \\
& \leq c_5(\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1)(N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p}.
\end{aligned}$$

Similarly, we also note $0 \leq \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \leq 1$ and Lemma 4 with $\beta = u$, $B = \|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2}$, $\varepsilon = c_5(N^*)^{-r-d}$, $\theta = d/(r+d)$, and $\tilde{L} = \lceil 1 + r/d \rceil$ imply

$$\begin{aligned}
I_1 & \leq c_5(N^*)^{-r-d} \left(\int_{\mathbb{I}^d} \left| \sum_{k=1}^{(N^*)^d} \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \right)^{1/p} \\
& = c_5(N^*)^{-r-d} \left(\int_{\mathbb{I}^d} \left| \sum_{k \neq k_x} \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \right)^{1/p} \\
& \quad + c_5(N^*)^{-r-d}.
\end{aligned}$$

The same approach as that in the proof of (43) yields that, for $0 < \tau \leq (N^*)^{-1-pr} \left(\frac{s}{2N^d}\right)$,

$$\left(\int_{\mathbb{I}^d} \left| \sum_{k \neq k_x} \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right|^p dx \right)^{1/p} \leq d^{1/p} 3^{2d} (N^*)^{-r} \left(\frac{s}{N^d}\right)^{\frac{1}{p}}.$$

Therefore,

$$I_1 \leq c_6(N^*)^{-r-d} \leq c_6(N^*)^{-r} \left(\frac{s}{N^d}\right)^{\frac{1}{p}},$$

where $c_6 := c_5(1 + d^{1/p} 3^{2d})$. Furthermore, by Lemma 2 that under $0 < \tau \leq \frac{s}{2N^d(N^*)^{1+pr}}$, we obtain

$$I_3 \leq c_2(N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p}.$$

Plugging the estimates of I_1, I_2, I_3 into (46), we then have

$$\|f - \mathcal{N}_{3,N^*,\tau}\|_{L^p(\mathbb{I}^d)} \leq C_4(N^*)^{-r} \left(\frac{s}{N^d}\right)^{1/p}$$

with $C_4 := c_2 + c_6 + c_5$. Thus, (14) holds.

Now we turn to the proof of (15). According to (45) and Lemma 3 with $\ell = 2$, $\theta = \frac{d}{2d+r}$, $\varepsilon = (N^*)^{-2d-r}$ and $\tilde{L} = \lceil 2 + r/d \rceil$, we have

$$\begin{aligned}
|\mathcal{N}_{3,N^*,\tau}(x)| & \leq \left| \sum_{k=1}^{(N^*)^d} h_{p_u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right| \\
& + c_5(N^*)^{-d-r} (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1).
\end{aligned}$$

But (12), together with $0 < \tau \leq 1/(2N^*)$, (40), $0 \leq \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \leq 1$ and (44) yields

$$\left| \sum_{k=1}^{(N^*)^d} h_{p_u,\xi_k,f}(x) \mathcal{N}_{1,N^*,\xi_k,\tau}(x) \right| \leq 3^d (\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1),$$

which implies (15) with $C_5 := (c_5 + 3^d)(\|f\|_{L^\infty(\mathbb{I}^d)} + c_1 d^{r/2} + 1)$. This completes the proof of Proposition 2. \blacksquare

C. Proof of Theorem 2

Let \mathbb{B} be a Banach space and V be a subset of \mathbb{B} . Denote by $\mathcal{N}(\varepsilon, V, \mathbb{B})$ the ε -covering number [17, Chap. 9] of V under the metric of \mathbb{B} , which is the minimal number of elements in an ε -net of V . The following lemma proved in [16, Theorem 1] gives rise to a tight estimate for the covering number of deep ReLU nets.

Lemma 5. *Let $\mathcal{H}_{n,L,\mathcal{R}}$ be defined by (17). Then*

$$\mathcal{N}(\varepsilon, \mathcal{H}_{n,L,\mathcal{R}}, L^\infty(\mathbb{I}^d)) \leq (c_7 \mathcal{R} D_{\max})^{3(L+1)^2 n} \varepsilon^{-n}, \tag{47}$$

where c_7 is a constant depending only on d and $D_{\max} = \max_{0 \leq \ell \leq L} d_\ell$.

To prove Theorem 2, we also need the following lemma, the proof of which can be found in [6].

Lemma 6. *Let \mathcal{H} be a collection of functions defined on \mathbb{I}^d and define*

$$f_{D,\mathcal{H}} = \arg \min_{f \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m (f(x_i) - y_i)^2. \tag{48}$$

Suppose there exist $n', \mathcal{U} > 0$, such that

$$\log \mathcal{N}(\varepsilon, \mathcal{H}, L^\infty(\mathbb{I}^d)) \leq n' \log \frac{\mathcal{U}}{\varepsilon}, \quad \forall \varepsilon > 0. \tag{49}$$

Then for any $h \in \mathcal{H}$ and $\varepsilon > 0$,

$$\begin{aligned}
& Pr\{\|\pi_M f_{D,\mathcal{H}} - f_\rho\|_\rho^2 > \varepsilon + 2\|h - f_\rho\|_\rho^2\} \\
& \leq \exp\left\{n' \log \frac{16\mathcal{U}M}{\varepsilon} - \frac{3m\varepsilon}{512M^2}\right\} \\
& \quad + \exp\left\{\frac{-3m\varepsilon^2}{16(3M + \|h\|_{L^\infty(\mathcal{X})})^2 (6\|h - f_\rho\|_\rho^2 + \varepsilon)}\right\}.
\end{aligned}$$

Now we are in a position to prove the upper bound of (24).

Proof of the upper bound of (24): For $N^* \geq \max\{4N, \tilde{C}\}$, Proposition 2 implies that there exists an $h_\rho \in \mathcal{H}_{L,n,\mathcal{R}}$ with L, n satisfying (16) and \mathcal{R} satisfying (18) such that

$$\begin{aligned}
\|f_\rho - h_\rho\|_\rho^2 & \leq D_{\rho X,p}^2 \|f_\rho - h_\rho\|_{L^p(\mathbb{I}^d)}^2 \\
& \leq C_4^2 D_{\rho X,p}^2 (N^*)^{-2r} \left(\frac{s}{N^d}\right)^{2/p} =: \mathcal{A}_p.
\end{aligned}$$

Recalling (15), we have

$$\|h_\rho\|_{L^\infty(\mathbb{I}^d)} \leq C_5.$$

But Lemma 5, together with the structure of deep nets in Proposition 2, (16) and (18), implies $D_{\max} \leq c_8 n$ with c_8 depending only on r and d and

$$\begin{aligned} \log \mathcal{N}(\varepsilon, \mathcal{H}_{n,L,\mathcal{R}}, L^\infty(\mathbb{I}^d)) &\leq c_9 L^2 n \log \frac{\mathcal{R}n}{\varepsilon} \\ &\leq c_{10} (N^*)^d \log \frac{N^* N^d}{s\varepsilon} \end{aligned}$$

for some positive constants c_9, c_{10} depending only on $d, r, C_1^*, c_7, c_8, \gamma, p$. Using the above three estimates in Lemma 6 with $n' = c_{10} (N^*)^d$, $\mathcal{U} = N^* N^d / s$, we have

$$\begin{aligned} &Pr\{\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon + 2\|h_\rho - f_\rho\|_\rho^2\} \\ &\leq \exp\left\{c_{10} (N^*)^d \log \frac{16MN^d N^*}{s\varepsilon} - \frac{3m\varepsilon}{512M^2}\right\} \\ &+ \exp\left\{\frac{-3m\varepsilon^2}{16(3M + C_5 + 1)^2 (6\mathcal{A}_p + \varepsilon)}\right\}. \end{aligned}$$

Thus, by scaling 3ε to ε , for $\varepsilon \geq \mathcal{A}_p$, we obtain

$$\begin{aligned} &Pr\{\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon\} \\ &\leq \exp\left\{c_{10} (N^*)^d \log \frac{48MN^d N^*}{s\varepsilon} - \frac{m\varepsilon}{512M^2}\right\} \\ &+ \exp\left\{\frac{-m\varepsilon^2}{16(3M + C_5 + 1)^2 (18\mathcal{A}_p + \varepsilon)}\right\}. \quad (50) \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbf{E}[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2] \\ &= \int_0^\infty Pr[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon] d\varepsilon \\ &= \int_{3\mathcal{A}_p}^\infty Pr[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon] d\varepsilon \\ &+ \int_0^{3\mathcal{A}_p} Pr[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon] d\varepsilon \\ &\leq \int_{3\mathcal{A}_p}^\infty Pr[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon] d\varepsilon + 3\mathcal{A}_p. \end{aligned}$$

From (50), we also have

$$\begin{aligned} &\int_{3\mathcal{A}_p}^\infty Pr[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2 > \varepsilon] d\varepsilon \\ &\leq \int_{3\mathcal{A}_p}^\infty \exp\left\{c_{10} (N^*)^d \log \frac{48MN^d N^*}{s\varepsilon} - \frac{m\varepsilon}{512M^2}\right\} d\varepsilon \\ &+ \int_{3\mathcal{A}_p}^\infty \exp\left\{\frac{-m\varepsilon^2}{16(3M + C_5 + 1)^2 (18\mathcal{A}_p + \varepsilon)}\right\} d\varepsilon \\ &=: J_1 + J_2. \end{aligned}$$

A direct computation then yields

$$\begin{aligned} J_2 &\leq \int_{3\mathcal{A}_p}^\infty \exp\left\{\frac{-m\varepsilon}{112(3M + C_5 + 1)^2}\right\} d\varepsilon \\ &\leq \frac{112(3M + C_5 + 1)^2}{m}. \end{aligned}$$

Set

$$(N^*)^{2r+d} \sim c_{11} m \left(\frac{s}{Nd}\right)^{\frac{2}{p}} / \log(c_{11}^{1/2r+d} m), \quad (51)$$

where $c_{11} := \frac{3C_4^2 D_{\rho_X,p}^2}{c_{10} c_{11} 1024M^2}$. It follows from (23) that $N^* \geq \max\{\tilde{C}, 4N\}$. Thus, for $p \geq 2$, we have, from the definition of \mathcal{A}_p , that

$$\begin{aligned} \log \frac{48MN^d N^*}{s\mathcal{A}_p} &\leq \log \frac{48MN^{dp} (N^*)^{2r+1}}{C_4^2 D_{\rho_X,p}^2 s^p} \\ &\leq \log \frac{48M (N^*)^{2r+1+dp}}{C_4^2 D_{\rho_X,p}^2} \leq c_{12} \log N^*, \end{aligned}$$

where $c_{12} := (2r + 1 + dp) \log\left(\frac{48M}{C_4^2 D_{\rho_X,p}^2} + 1\right)$. Then

$$c_{10} c_{12} (N^*)^d \log N^* \leq \frac{3m\mathcal{A}_p}{1024M^2},$$

which implies

$$J_1 \leq \int_{3\mathcal{A}_p}^\infty \exp\left\{\frac{-m\varepsilon}{1024M^2}\right\} d\varepsilon \leq \frac{1024M^2}{m}.$$

Thus,

$$\mathbf{E}[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2] \leq \frac{c_{13}}{m} + 3\mathcal{A}_p,$$

where $c_{13} := 1024M^2 + 112(3M + C_5 + 1)^2$. Hence,

$$\begin{aligned} &\mathbf{E}[\|\pi_M f_{D,n,L} - f_\rho\|_\rho^2] \\ &\leq C_7 \left(\frac{m}{\log m}\right)^{\frac{-2r}{2r+d}} \left(\frac{s}{Nd}\right)^{\frac{2}{p} - \frac{2r}{2r+d}}. \end{aligned}$$

This provides the upper bound of (24). \blacksquare

V. PROOF OF THE LOWER BOUNDS

In this section, we present a general lower bound estimate for Theorem 1 and Theorem 2. To this end, we need the following assumption for the qualification of the distribution ρ .

Assumption 1. Assume

- (A) $f_\rho \in Lip^{(N,s,r,c_0)}$.
- (B) ρ_X is the uniform distribution on \mathbb{I}^d .
- (C) $y = f_\rho(x) + \nu$, where ν and x are independent and ν is drawn according to the standard normal distribution $\mathcal{N}(0, 1)$.

Let $\mathcal{M}_1(N, s, r, c_0)$ be the set of all distributions that satisfy Assumption 1 and Ψ_m be the set of estimators f_D derived from D_m . Then

$$\begin{aligned} &\sup_{\rho \in \mathcal{M}(N,s,r,c_0)} \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho\|_\rho^2] \\ &\geq \sup_{\rho \in \mathcal{M}_1(N,s,r,c_0)} \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho\|_\rho^2]. \quad (52) \end{aligned}$$

The following theorem is a more general lower bound than that in Theorem 1.

Theorem 3. If m satisfies (3), then there exists a constant \tilde{C} independent of m, s or N , such that

$$\sup_{\rho \in \mathcal{M}(N,s,r,c_0)} \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho\|_\rho^2] \geq \tilde{C} m^{-\frac{2r}{2r+d}} \left(\frac{s}{Nd}\right)^{\frac{d}{2r+d}}. \quad (53)$$

It is easy to see that the lower bound of Theorem 1 is a direct consequence of Theorem 3. Before presenting the proof, we introduce a function g that satisfies the following assumption.

Assumption 2. Assume that $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies $\text{supp}(g) = [-1/(2\sqrt{d}), 1/(2\sqrt{d})]^d$, $g(x) = 1$ for $x \in [-1/(4\sqrt{d}), 1/(4\sqrt{d})]^d$ and $g \in \text{Lip}^{(r, c_0 2^{v-1})}$, where $\text{supp}(g)$ denotes the support of g .

Let $\{\epsilon_k\}_{k=1}^{(N^*)^d}$ be a set of independent Rademacher random variables, i.e.,

$$\Pr(\epsilon_k = 1) = \Pr(\epsilon_k = -1) = \frac{1}{2}, \quad \forall k = 1, 2, \dots, (N^*)^d. \quad (54)$$

For $x \in \mathbb{I}^d$, define

$$g_k(x) := (N^*)^{-r} g(N^*(x - \xi_k)), \quad (55)$$

where ξ_k is the center of the cube B_k . Since

$$\|N^*(x - \xi_k) - N^*(x - \xi_{k'})\| = N^* \|\xi_k - \xi_{k'}\| \geq 1, \quad \forall k \neq k',$$

at least one of $N^*(x - \xi_k)$ and $N^*(x - \xi_{k'})$ lies outside $(-1/(2\sqrt{d}), 1/(2\sqrt{d}))^d$. Then it follows from Assumption 2 that

$$g_k(x) = 0, \quad \text{if } x \notin \dot{B}_k, \quad (56)$$

where $\dot{B}_k = B_k \setminus \partial B_k$ and ∂A denotes the boundary of a cube A .

Given $S = \cup_{j \in \Lambda_s} A_j$, consider the set \mathcal{F}_{S, N^*} of all functions,

$$f(x) = \begin{cases} \sum_{k=1}^{(N^*)^d} \epsilon_k g_k(x), & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases}$$

with ϵ_k that satisfies (54). It is obvious that \mathcal{F}_{S, N^*} is a set of random functions. The following lemma shows that it is almost surely a subset of $\text{Lip}^{(N, s, r, c_0)}$.

Lemma 7. If g_k is defined by (55) with g satisfying the Assumption 2, then for $N^* \in \mathbb{N}$ and $S = \cup_{j \in \Lambda_s} A_j$, then

$$\mathcal{F}_{S, N^*} \subset \text{Lip}^{(N, s, r, c_0)}$$

almost surely.

Proof: From the definition of \mathcal{F}_{S, N^*} , it is obvious that each $f \in \mathcal{F}_{S, N^*}$ is s -sparse in N^d partitions. So, it suffices to prove that $f \in \mathcal{F}_{S, N^*}$ implies $f \in \text{Lip}^{(r, c_0)}$ almost surely. For $x, x' \in \mathbb{I}^d$ with $x \neq x'$, we divide the proof into four cases: $x, x' \in S$, $x \in S$ but $x' \notin S$, $x \notin S$ but $x' \in S$ and $x, x' \notin S$.

Case 1: $x, x' \in S$. If $x, x' \in B_{k_0} \cap S$ for some $k_0 \in \{1, \dots, (N^*)^d\}$, then (56) yields $(g_k)_\alpha^{(u)}(x) = 0$ for $k \neq k_0$. So, for each $f \in \mathcal{F}_{S, N^*}$, we get from $|\epsilon_k| = 1$, (56), (55) and $0 < v \leq 1$ that

$$\begin{aligned} & |f_\alpha^{(u)}(x) - f_\alpha^{(u)}(x')| \\ &= \left| \sum_{k=1}^{(N^*)^d} \epsilon_k (g_k)_\alpha^{(u)}(x) - (g_k)_\alpha^{(u)}(x') \right| \\ &= \left| (g_{k_0})_\alpha^{(u)}(x) - (g_{k_0})_0^{(u)}(x') \right| \\ &\leq (N^*)^{-r+u} \left| g_\alpha^{(u)}(N^*(x - \xi_{k_0})) - g_\alpha^{(u)}(N^*(x' - \xi_{k_0})) \right| \\ &\leq c_0 2^{v-1} \|x - x'\|^v \leq c_0 \|x - x'\|^v. \end{aligned}$$

If $x \in B_{k_1} \cap S$ but $x' \in B_{k_2} \cap S$ for some $k_1, k_2 \in \{1, \dots, (N^*)^d\}$ with $k_1 \neq k_2$, we can choose $z \in \partial B_{k_1}$ and

$z' \in \partial B_{k_2}$ such that z, z' are on the segment between x and x' . Then

$$\|x - z\| + \|x' - z'\| \leq \|x - x'\|. \quad (57)$$

So, Assumption 2, (56), $0 < v \leq 1$, Jensen's inequality and (57) show

$$\begin{aligned} & |f_\alpha^{(u)}(x) - f_\alpha^{(u)}(x')| \\ &= \left| \sum_{k=1}^{(N^*)^d} \epsilon_k [(g_k)_\alpha^{(u)}(x) - (g_k)_\alpha^{(u)}(x')] \right| \\ &\leq \left| (g_{k_1})_\alpha^{(u)}(x) \right| + \left| (g_{k_2})_\alpha^{(u)}(x') \right| \\ &= \left| (g_{k_1})_\alpha^{(u)}(x) - (g_{k_1})_\alpha^{(u)}(z) \right| + \left| (g_{k_2})_\alpha^{(u)}(x') - (g_{k_2})_\alpha^{(u)}(z') \right| \\ &\leq (N^*)^{-r+u} \left[\left| g_\alpha^{(u)}(N^*(x - \xi_{k_1})) - g_\alpha^{(u)}(N^*(z - \xi_{k_1})) \right| \right. \\ &\quad \left. + \left| g_\alpha^{(u)}(N^*(x' - \xi_{k_2})) - g_\alpha^{(u)}(N^*(z' - \xi_{k_2})) \right| \right] \\ &\leq c_0 2^v \left[\frac{\|x - z\|^v}{2} + \frac{\|x' - z'\|^v}{2} \right] \\ &\leq c_0 2^v \left[\frac{\|x - z\|}{2} + \frac{\|x' - z'\|}{2} \right]^v \leq c_0 \|x - x'\|^v. \end{aligned}$$

These two assertions imply that $f \in \text{Lip}^{(r, c_0)}$ almost surely and proves Lemma 7 for the first case.

Case 2: Suppose $x \in S$, $x' \notin S$. There is some $k_3 \in \{1, \dots, (N^*)^d\}$ such that $x \in S \cap B_{k_3}$. For each $f \in \mathcal{F}_{S, N^*}$ and any $x' \notin S$, it follows from (55) and Assumption 2 that

$$f(x') = 0 = f(z), \quad \forall z \in \partial B_{k_3}.$$

Select a $z'' \in \partial B_{k_3}$ on the segment between x and x' . Then, $\|x - x'\| \geq \|x - z''\|$. Hence, the result in the first case above shows that

$$\begin{aligned} & |f_\alpha^{(u)}(x) - f_\alpha^{(u)}(x')| = |f_\alpha^{(u)}(x) - f_\alpha^{(u)}(z'')| \\ &\leq c_0 \|x - z''\|^v \leq c_0 \|x - x'\|^v. \end{aligned}$$

Case 3: Suppose $x' \in S$, $x \notin S$. The proof of this case is the same as that of Case 2.

Case 4: Suppose $x, x' \notin S$. For each $f \in \mathcal{F}_{S, N^*}$ and any $x, x' \notin S$, we have

$$|f_\alpha^{(u)}(x) - f_\alpha^{(u)}(x')| = 0 \leq c_0 \|x - x'\|^v.$$

Combining the above four cases, we complete the proof of Lemma 7. \blacksquare

Let \mathcal{H}_{S, N^*} be the set of all functions

$$h(x) = \begin{cases} \sum_{k=1}^{(N^*)^d} c_k g_k(x), & \text{if } x \in S, \\ 0, & \text{otherwise} \end{cases}$$

with $c_k \in \mathbb{R}$. It then follows from the definition of \mathcal{F}_{S, N^*} that

$$\mathcal{F}_{S, N^*} \subset \mathcal{H}_{S, N^*}. \quad (58)$$

The following lemma constructs an orthonormal basis of \mathcal{H}_{S, N^*} .

Lemma 8. Let \mathcal{H}_{S, N^*} be defined as above with g_k and g that satisfy (55) and Assumption 2, respectively. Let

$$g_{k, S}^*(x) := \begin{cases} g_k(x), & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases} \quad (59)$$

Then, the system $\left\{ \frac{g_{k,S}^*(\cdot)}{\|g_{k,S}^*\|_\rho} : k = 1, \dots, (N^*)^d \right\}$ is an orthonormal basis of \mathcal{H}_{S,N^*} using the inner product of $L^2_{\rho_X}$.

Proof: For $k \neq k'$, it follows from (56) and (59) that

$$\int_{\mathbb{I}^d} g_{k,S}^*(x) g_{k',S}^*(x) d\rho_X = \int_S g_k(x) g_{k'}(x) d\rho_X = 0. \quad (60)$$

Therefore, $\{g_{k,S}^*(\cdot) : k = 1, \dots, (N^*)^d\}$ is an orthogonal set in $L^2_{\rho_X}$. Noting further $\|g_{k,S}^*\|_\rho \neq 0$ for all $k \in \{1, \dots, (N^*)^d\}$,

$$\int_{\mathbb{I}^d} \left(\frac{g_{k,S}^*(x)}{\|g_{k,S}^*\|_\rho} \right)^2 d\rho_X = 1, \quad \forall k = 1, 2, \dots, (N^*)^d$$

and \mathcal{H}_{S,N^*} is an $(N^*)^d$ -dimensional linear space, we may conclude that the system $\left\{ \frac{g_{k,S}^*(\cdot)}{\|g_{k,S}^*\|_\rho} : k = 1, \dots, (N^*)^d \right\}$ is an orthonormal basis of \mathcal{H}_{S,N^*} . This completes the proof of Lemma 8. ■

To prove the lower bound, we need the following three lemmas. The first one can be found in [17, Lemma 3.2].

Lemma 9. *Let U be an ℓ -dimensional real vector; θ a zero-mean random variable with range $\{-1, 1\}$, and ν an ℓ -dimensional random vector of standard normal variable, independent of U . Denote*

$$\psi := \theta U + \nu.$$

Then there exists an absolute constant $\tilde{C}_1 > 0$ such that

$$\min_{f^* : \mathbb{R}^\ell \rightarrow \{-1, 1\}} \Pr\{f^*(\psi) \neq \theta\} \geq \tilde{C}_1 e^{-\|U\|_\ell^2/2},$$

where $\|\cdot\|_\ell$ denotes the ℓ -dimensional Euclidean norm and the minimization is over all functions $f^* : \mathbb{R}^\ell \rightarrow \{-1, 1\}$.

Lemma 10. *Under (B) in Assumption 1, if g satisfies Assumption 2, then for any $k \in \{1, \dots, (N^*)^d\}$*

$$\int_{B_k} [g(N^*(x - \xi_k))]^2 d\rho_X \geq \tilde{C}_2 (N^*)^{-d}, \quad (61)$$

where the constant \tilde{C}_2 dependent only on d .

Proof: It follows from Assumption 2 and (B) that

$$\begin{aligned} & \int_{B_k} [g(N^*(x - \xi_k))]^2 d\rho_X \\ & \geq \int_{B_k} [g(N^*(x - \xi_k))]^2 dx \\ & \geq (N^*)^{-d} \int_{[-1/(2\sqrt{d}), 1/(2\sqrt{d})]^d} |g(x)|^2 dx \\ & \geq (N^*)^{-d} \int_{[-1/(4\sqrt{d}), 1/(4\sqrt{d})]^d} dx \\ & = (2\sqrt{d}N^*)^{-d}, \end{aligned} \quad (62)$$

where the second inequality holds since $N^*(x - \xi_k)$ is restricted to some subset of \mathbb{R}^d that contains $[-1/(2\sqrt{d}), 1/(2\sqrt{d})]^d$ for $x \in B_k$. This completes the proof of Lemma 10 with $\tilde{C}_2 = \tilde{C}_3(2\sqrt{d})^{-d}$. ■

If $N^* \geq 4N$, noting that $\{A_j\}_{j=1}^{N^d}$ and $\{B_k\}_{k=1}^{(N^*)^d}$ are cubic partitions of \mathbb{I}^d , we may conclude that each A_j then contains at least $\left(\frac{N^*}{N} - 2\right)^d \geq \left(\frac{N^*}{2N}\right)^d B_k$'s. For each $j \in \Lambda_s$, denote

$$\Lambda_j^* := \{k \in \{1, \dots, (N^*)^d\} : B_k \subseteq A_j\}. \quad (63)$$

Then

$$|\Lambda_j^*| \geq \left(\frac{N^*}{2N}\right)^d. \quad (64)$$

With the above preparations, we present the following lemma, which will play a crucial role in our analysis.

Lemma 11. *Let $D_m = \{(x_i, y_i)\}_{i=1}^m$ be the set of samples which are independently drawn according to some distribution ρ with the marginal distribution ρ_X satisfying (B) and $y_i = f_\rho(x_i) + \nu_i$, where $f_\rho \in \mathcal{F}_{S,N^*}$ and ν_i is the standard normal variable. If $N^* \geq 4N$ and $N^* = \left\lceil \left(\frac{ms}{N^d}\right)^{\frac{1}{2r+d}} \right\rceil$. Then for any $j \in \Lambda_s$, $k \in \Lambda_j^*$, there exists a constant \tilde{C}_3 independent of m , s , N or N^* such that*

$$\min_{h: \mathbb{R}^m \rightarrow \{-1, 1\}} \Pr\{h((y_1, \dots, y_m)) \neq \epsilon_k\} \geq \tilde{C}_3 > 0. \quad (65)$$

Proof: Write $D_{in} = \{x_i\}_{i=1}^m$ and $D_{in,S} := D_{in} \cap S$. For each $j \in \Lambda_s$ and $k \in \Lambda_j^*$, denote further $B_{k,D} := B_k \cap D_{in} := \{x_{i,k}\}_{i=1}^{\ell'}$, where $\ell' = 0$ means $B_{k,D} = \emptyset$. We then divide the proof into the following three steps.

Step 1: Estimating $|D_{in,S}|$. Since ρ_X is the uniform distribution on \mathbb{I}^d , for each $x_i \in D_{in}$,

$$\Pr\{x_i \in S\} = \frac{s}{N^d}.$$

For $i = 1, \dots, m$, define

$$V_i := \mathcal{I}_{x_i \in S} := \begin{cases} 1, & \text{with probability } \frac{s}{N^d} \\ 0, & \text{with probability } 1 - \frac{s}{N^d}. \end{cases}$$

Then

$$|D_{in,S}| = \sum_{i=1}^m V_i = \sum_{i=1}^m \mathcal{I}_{x_i \in S}.$$

This implies

$$\begin{aligned} \mathbf{E}\{|D_{in,S}|\} &= \sum_{i=1}^m \mathbf{E}\{\mathcal{I}_{x_i \in S}\} = \sum_{i=1}^m \Pr\{x_i \in S\} \\ &= \sum_{i=1}^m \frac{s}{N^d} = \frac{ms}{N^d}. \end{aligned}$$

So, it follows from Markov's inequality that

$$\Pr\left\{|D_{in,S}| > \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \leq \frac{N^d \mathbf{E}\{|D_{in,S}|\}}{2ms} = \frac{1}{2}.$$

The above estimate, together with the formula of total probability, implies that

$$\begin{aligned}
& \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\{h(y_D) \neq \epsilon_k\} \quad (66) \\
&= \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| > \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \\
&\quad \Pr\left\{|D_{in,S}| > \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \\
&+ \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \\
&\quad \Pr\left\{|D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \\
&\geq \frac{1}{2} \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil\right\},
\end{aligned}$$

where $h(y_D) := h((y_1, \dots, y_m))$.

Step 2: Estimating the conditional probability. If \mathcal{A} and \mathcal{B} are random events, then

$$\Pr\{\mathcal{A}\} = \mathbf{E}\{\mathcal{I}_{\mathcal{A}}\} = \mathbf{E}\{\mathbf{E}\{\mathcal{I}_{\mathcal{A}}|\mathcal{B}\}\} = \mathbf{E}\{\Pr\{\mathcal{A}|\mathcal{B}\}\}, \quad (67)$$

where $\mathcal{I}_{\mathcal{A}}$ denotes the indicator of the event \mathcal{A} . Hence,

$$\begin{aligned}
& \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil\right\} \\
&= \mathbf{E}\left\{\min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\{h(y_D) \neq \epsilon_k \mid \right. \\
&\quad \left. |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\}\right\}. \quad (68)
\end{aligned}$$

For each $j \in \Lambda_s$ and $k \in \Lambda_j^*$, it follows from (63) that $\ell' = |B_{k,D}| \leq |D_{in,S}|$. Then for each $h: \mathbb{R}^m \rightarrow \{-1,1\}$, from the formula of total probability again, we obtain

$$\begin{aligned}
& \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\right\} \quad (69) \\
&= \sum_{\ell=0}^{\left\lceil \frac{2ms}{N^d} \right\rceil} \Pr\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, \\
&\quad D_{in}, \ell' = \ell\} \Pr\left\{\ell' = \ell \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\right\}
\end{aligned}$$

and

$$\sum_{\ell=0}^{\left\lceil \frac{2ms}{N^d} \right\rceil} \Pr\left\{\ell' = \ell \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\right\} = 1. \quad (70)$$

Given D_{in} , $\ell' = 0$, $|D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil$, for each $k \in \{1, \dots, (N^*)^d\}$, it follows from the definition of \mathcal{F}_{S,N^*} and (56) that there exists some $k' \neq k$ such that

$$y_i = \sum_{k=1}^{(N^*)^d} \epsilon_k g_k(x_i) + \nu_i = \epsilon_{k'} g_{k'}(x_i) + \nu_i, \quad i = 1, 2, \dots, m,$$

which is independent of ϵ_k . That is, ϵ_k is independent of (y_1, \dots, y_m) . Thus, it follows from (54) that

$$\min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\{h((y_1, \dots, y_m)) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}, \ell' = 0\} = \frac{1}{2}. \quad (71)$$

Given D_{in} , $|D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil$ and $\ell' = \ell$ with $\ell \geq 1$, for each $j \in \Lambda_s$ and $k \in \Lambda_j^*$, then we get from the definition of \mathcal{F}_{S,N^*} and (56) that there exists a $k' \neq k$, such that

$$y_i = \sum_{k=1}^{(N^*)^d} \epsilon_k g_k(x_i) + \nu_i = \epsilon_{k'} g_{k'}(x_i) + \nu_i, \quad x_i \in D_{in} \setminus B_{k,D}$$

which is independent of ϵ_k . Write

$$y_{i,k} = \sum_{k=1}^{(N^*)^d} \epsilon_k g_k(x_{i,k}) + \nu_i = \epsilon_k g_k(x_{i,k}) + \nu_i, \quad i = 1, \dots, \ell'.$$

Then, there exists an $h^*: \mathbb{R}^{\ell'} \rightarrow \{-1,1\}$, such that

$$\begin{aligned}
& \Pr\left\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}, \ell' = \ell\right\} \quad (72) \\
&= \Pr\left\{h^*(y_{D,\ell'}) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}, \ell' = \ell\right\},
\end{aligned}$$

where $y_{D,\ell'} := (y_{1,k}, \dots, y_{\ell',k})$. From (56) again, it is easy to see that

$$\begin{aligned}
& (y_{1,k}, \dots, y_{\ell',k}) \quad (73) \\
&:= \epsilon_k (g_k(x_{1,k}), \dots, g_k(x_{\ell',k})) + (\nu_{1,k}, \dots, \nu_{\ell',k}),
\end{aligned}$$

Therefore, applying Lemma 9 with $U = (g_k(x_{1,k}), \dots, g_k(x_{\ell',k}))$ and $\theta = \epsilon_k$, we get from (73) and (72) that for each $k \in \Lambda_j^*$ and $j \in \Lambda_s$,

$$\begin{aligned}
& \min_{h^*: \mathbb{R}^{\ell'} \rightarrow \{-1,1\}} \Pr\left\{h^*(y_{D,\ell'}) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, \right. \\
&\quad \left. D_{in}, \ell' = \ell\right\} \\
&\geq \tilde{C}_1 \exp\left(-\frac{(g_k(x_{1,k}))^2 + \dots + (g_k(x_{\ell',k}))^2)}{2}\right). \quad (74)
\end{aligned}$$

Putting (74) and (71) into (69) and noting (72) and (70), we obtain that for each $k \in \Lambda_j^*$ and $j \in \Lambda_s$,

$$\begin{aligned}
& \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} \Pr\{h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, \\
&\quad D_{in}\} \geq \frac{1}{2} \Pr\left\{\ell' = 0 \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\right\} \\
&+ \tilde{C}_1 \sum_{\ell=1}^{\left\lceil \frac{2ms}{N^d} \right\rceil} \exp\left(-\frac{(g_k(x_{1,k}))^2 + \dots + (g_k(x_{\ell,k}))^2)}{2}\right) \\
&\quad \Pr\left\{\ell' = \ell \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil, D_{in}\right\} \\
&\geq \min\left\{\frac{1}{2}, \tilde{C}_1 \mathcal{B}(m, s, N, g_k)\right\}. \quad (75)
\end{aligned}$$

where

$$\mathcal{B}(m, s, N, g_k) := \exp\left(-\frac{\sum_{x_i \in D_{in,S}, |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil} (g_k(x_i))^2}{2}\right).$$

Step 3: Estimating the probability. Putting (75) into (68), we have, from Jensen's inequality with the convexity of $\exp(-\cdot)$, that

$$\begin{aligned} & \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} Pr \left\{ h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil \right\} \\ & \geq \mathbf{E} \left\{ \min \left\{ \frac{1}{2}, \tilde{C}_1 \mathcal{B}(m, s, N, g_k) \right\} \right\} \\ & \geq \min \left\{ \frac{1}{2}, \tilde{C}_1 \mathcal{C}(m, s, N, g_k) \right\}, \end{aligned}$$

where

$$\begin{aligned} & \mathcal{C}(m, s, N, g_k) \\ & := \exp \left(- \frac{\mathbf{E} \left\{ \sum_{x_i \in D_{in,S}, |D_{in,S}| \leq \lceil \frac{2ms}{N^d} \rceil} (g_k(x_i))^2 \right\}}{2} \right). \end{aligned}$$

But (61) implies that for each $j \in \Lambda_s$ and $k \in \Lambda_j^*$,

$$\begin{aligned} & \int_{\mathbb{I}^d} g_k^2(x) d\rho_X = \int_{B_k} g_k^2(x) d\rho_X \quad (76) \\ & = (N^*)^{-2r} \int_{B_k} [g(N^*(x - \xi_k))]^2 d\rho_X \geq \tilde{C}_2 (N^*)^{-2r-d}, \end{aligned}$$

which yields

$$\mathbf{E} \left\{ \sum_{x_i \in D_{in,S}, |D_{in,S}| \leq \lceil \frac{2ms}{N^d} \rceil} (g_k(x_i))^2 \right\} \geq \tilde{C}_2 (N^*)^{-2r-d} \frac{2ms}{N^d}.$$

Therefore, for each $j \in \Lambda_s$ and $k \in \Lambda_j^*$

$$\begin{aligned} & \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} Pr \left\{ h(y_D) \neq \epsilon_k \mid |D_{in,S}| \leq \left\lceil \frac{2ms}{N^d} \right\rceil \right\} \\ & \geq \min \left\{ \frac{1}{2}, \tilde{C}_1 \exp \left(- \frac{\tilde{C}_2 (N^*)^{-2r-d} \frac{2ms}{N^d}}{2} \right) \right\}. \end{aligned}$$

Inserting the above estimate into (66), we then have

$$\begin{aligned} & \min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} Pr \{ h(y_D) \neq \epsilon_k \} \\ & \geq \frac{1}{2} \min \left\{ \frac{1}{2}, \tilde{C}_1 \exp \left(- \frac{\tilde{C}_2 (N^*)^{-2r-d} \frac{2ms}{N^d}}{2} \right) \right\}. \end{aligned}$$

Since $N^* = \left\lceil \left(\frac{ms}{N^d} \right)^{1/(2r+d)} \right\rceil$, we see that, for any $k \in \Lambda_j^*$, $j \in \Lambda_s$,

$$\min_{h: \mathbb{R}^m \rightarrow \{-1,1\}} Pr \{ h((y_1, \dots, y_m)) \neq \epsilon_k \} \geq \tilde{C}_3$$

with $\tilde{C}_3 = \frac{1}{2} \min\{1/2, \tilde{C}_1 e^{-\tilde{C}_2/2}\}$. This completes the proof of Lemma 11. \blacksquare

We are now in a position to prove our main result.

Proof of Theorem 3. For $f_D \in \Psi_m$, define

$$\begin{aligned} \hat{f}_D(x) & := \sum_{k=1}^{(N^*)^d} \frac{\int_{\mathbb{I}^d} f_D(x) g_{k,S}^*(x) d\rho_X}{\|g_{k,S}^*\|_\rho} g_{k,S}^*(x) \\ & =: \sum_{k=1}^{(N^*)^d} \hat{\epsilon}_k g_{k,S}^*(x), \quad (77) \end{aligned}$$

where $g_{k,S}^*$ is defined by (59). In view of Lemma 8, we observe that \hat{f}_D is the orthogonal projection of f_D to \mathcal{H}_{S,N^*} . For $N^* \geq 4N$ and $f_\rho^\epsilon \in \mathcal{F}_{S,N^*} \subset \mathcal{H}_{S,N^*}$ with $\epsilon = (\epsilon_1, \dots, \epsilon_{(N^*)^d})$ and ϵ_k the Rademacher random variable, it then follows from (58), (59), (56) and (63) that

$$\begin{aligned} & \|f_D - f_\rho^\epsilon\|_\rho^2 \geq \|\hat{f}_D - f_\rho^\epsilon\|_\rho^2 \\ & \geq \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} \int_{B_{k'}} [f_D(x) - f_\rho^\epsilon(x)]^2 d\rho_X \\ & = \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} \int_{B_{k'}} \left[\sum_{k=1}^{(N^*)^d} (\hat{\epsilon}_k - \epsilon_k) g_k(x) \right]^2 d\rho_X \\ & = \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} \int_{B_{k'}} [\hat{\epsilon}_{k'} - \epsilon_{k'}]^2 [g_{k'}(x)]^2 d\rho_X \\ & = (N^*)^{-2r} \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} [\hat{\epsilon}_{k'} - \epsilon_{k'}]^2 \\ & \quad \times \int_{B_{k'}} [g(N^*(x - \xi_{k'}))]^2 d\rho_X. \end{aligned}$$

Define $\tilde{\epsilon}_k = \begin{cases} 1, & \hat{\epsilon}_k \geq 0 \\ -1, & \hat{\epsilon}_k < 0. \end{cases}$ Noting that $\tilde{\epsilon}_k$ is a decision of ϵ_k based on D , we may conclude that there exists some $h_k: \mathbb{R}^m \rightarrow \{-1,1\}$ such that $h_k(y_1, \dots, y_m) = \tilde{\epsilon}_k$. Since $|\hat{\epsilon}_k - \epsilon_k| \geq \frac{|\tilde{\epsilon}_k - \epsilon_k|}{2}$, we have from Lemma 10 that

$$\begin{aligned} & \|f_D - f_\rho^\epsilon\|_\rho^2 \geq \frac{\tilde{C}_2}{4} (N^*)^{-2r-d} \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} [\tilde{\epsilon}_{k'} - \epsilon_{k'}]^2 \\ & \geq \frac{\tilde{C}_2}{4} (N^*)^{-2r-d} \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} \mathcal{I}_{\tilde{\epsilon}_{k'} \neq \epsilon_{k'}}. \end{aligned}$$

Hence,

$$\begin{aligned} & \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho^\epsilon\|_\rho^2] \\ & \geq \frac{\tilde{C}_2}{4} (N^*)^{-2r-d} \inf_{\tilde{\epsilon} = (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{(N^*)^d})} \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} Pr\{\tilde{\epsilon}_{k'} \neq \epsilon_{k'}\}. \end{aligned}$$

But Lemma 11 and (64) assure that for any set of independent Rademacher random variables $\epsilon = (\epsilon_1, \dots, \epsilon_{(N^*)^d})$,

$$\begin{aligned} & \inf_{\tilde{\epsilon}} \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} Pr\{\tilde{\epsilon}_{k'} \neq \epsilon_{k'}\} \\ & = \sum_{j \in \Lambda_s} \sum_{k' \in \Lambda_j^*} \inf_{\tilde{\epsilon}_{k'}} Pr\{\tilde{\epsilon}_{k'} \neq \epsilon_{k'}\} \geq \tilde{C}_3 s \left(\frac{N^*}{2N} \right)^d. \end{aligned}$$

Therefore, Lemma 7, together with (52), yields

$$\begin{aligned} & \sup_{\rho \in \mathcal{M}(N,s,r)} \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho\|_\rho^2] \\ & \geq \sup_{\tilde{\epsilon}} \inf_{f_D \in \Psi_m} \mathbf{E}[\|f_D - f_\rho^\epsilon\|_\rho^2] \\ & \geq \frac{\tilde{C}_2}{4} (N^*)^{-2r-d} \tilde{C}_3 s \left(\frac{N^*}{2N} \right)^d = \frac{\tilde{C}_2 \tilde{C}_3}{2^{d+2}} (N^*)^{-2r} \frac{s}{N^d}. \end{aligned}$$

By setting $N^* = \left\lceil \left(\frac{ms}{Nd}\right)^{1/(2r+d)} \right\rceil$, (3) implies $N^* \geq 4N$. Hence,,

$$\sup_{\rho \in \mathcal{M}(N, s, r, c_0)} \inf_{f_D \in \Psi_m} \mathbb{E}[\|f_D - f_\rho\|_\rho^2] \geq \tilde{C} m^{-\frac{2r}{2r+d}} \left(\frac{s}{Nd}\right)^{\frac{d}{2r+d}},$$

where $\tilde{C} := \frac{\tilde{C}_2 \tilde{C}_3}{2^{d+2}}$. This completes the proof of Theorem 3. \square

Proof of Theorem 2: The upper bound of (24) was established in Section IV. The lower bound of (24) is a direct corollary of Theorem 3. This completes the proof of Theorem 2. \blacksquare

Proof of Theorem 1: The upper bound of (4) can be derived from (24) with $p = 2$ and the lower bound is a consequence of Theorem 3. This completes the proof of Theorem 1. \blacksquare

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