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Sliding mode control of switched hybrid systems with stochastic perturbation☆

Ligang Wu [a,*], Daniel W.C. Ho [b], C.W. Li [b]

[a] *Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, 150001, PR China*

[b] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*

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ABSTRACT

This paper is concerned with the sliding mode control (SMC) of a continuous-time switched stochastic system. A sufficient condition for the existence of reduced-order sliding mode dynamics is derived and an explicit parametrization of the desired sliding surface is also given. Then, a sliding mode controller is then synthesized for reaching motion. Moreover, the observer-based SMC problem is also investigated. Some sufficient conditions are established for the existence and the solvability of the desired observer and the observer-based sliding mode controller is synthesized. Finally, numerical examples are provided to illustrate the effectiveness of the proposed theory.

1. Introduction

Switched systems are considered to be a class of hybrid systems which consist of a family of subsystems described by continuous-time (or discrete-time) dynamics and these subsystems are governed by a switching signal [1,2]. Switched systems have received increasing attention over the past few years. It is known that many real-world systems can be modeled as switched systems, for example, chemical process systems, transportation systems, computer controlled systems and communication systems. In particular, many intelligent control strategies are designed based on the idea of switching controllers to overcome the shortcoming of using a single controller. As a result, the overall performance is improved [3]. In fact, a large number of papers have been reported recently involving such switching systems. In particular, stability and stabilization problems were investigated in [1,4]; a filtering problem was studied in [5]; optimal performance analysis and control problems were considered in [6]; and a model reduction problem was addressed in [7,8].

Recently, there has been an enormous growth of interest in using the dwell time approach to deal with switched systems [9]. Given a positive constant τ_d called 'dwell time' and let $\mathscr{S}(\tau_d)$ denote the set of all switching signals with interval between consecutive discontinuities no smaller than τ_d, it has been shown that one can pick τ_d sufficiently large such that the switched system considered is exponentially stable for any switching signal belonging to $\mathscr{S}(\tau_d)$. Hespanha and Morse [10] extended this concept to develop the 'average dwell time' approach subsequently, which means that the average time interval between consecutive switchings is no less than a specified constant τ_d^*, and they proved that if such a constant τ_d^* is sufficiently large, then the switched system is exponentially stable. Some extended results on the average dwell time approach to the switched systems can be refereed to in [11,8,6] and references therein.

On the other hand, the study of stochastic systems has been of great interest in many branches of science and engineering applications. For instance, Lu et al. investigated the robust stability and the robust stabilization of uncertain stochastic systems with time-varying delays by using the linear matrix inequality (LMI) approach [12]. Wang et al. studied the stochastic stabilization for a class of bilinear continuous time-delay uncertain systems with Markovian jumping parameters [13]. Xu and Chen designed a robust $\mathscr{H}_\infty$ controller for uncertain stochastic systems with state delay [14]. Niu et al. designed an integral switching function and synthesized a sliding mode controller for uncertain stochastic systems with state delay [15]. Xu and Chen [16] developed an $\mathscr{H}_\infty$ output feedback control for uncertain stochastic systems with time-varying delays.

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* Corresponding author.
E-mail addresses: ligangwu@hit.edu.cn (L. Wu), madaniel@cityu.edu.hk (D.W.C. Ho).

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Since the 1950's, sliding mode control (SMC) has proven to be an effective robust control strategy for nonlinear systems and incompletely modeled systems. In the past two decades, SMC has been successfully applied to a wide variety of practical engineering systems such as robot manipulators, aircraft, underwater vehicles, spacecraft, flexible space structures, electrical motors, power systems, and automotive engines [15]. Basically, the idea of SMC is to utilize a discontinuous control to force the system state trajectories to some predefined sliding surfaces on which the system has desired properties such as stability, disturbance rejection capability, and tracking ability [11,17]. Many important results have been reported for this kind of control strategy. In particular, SMC has been investigated for uncertain systems [18], time-delay systems [19], stochastic systems [15], Markovian jump systems [20], switched hybrid time-delay systems [11], and singular systems [17].

Recently, Shi et al. [20] studied the SMC of Markovian jump systems. Niu et al. [15] investigated SMC of the uncertain stochastic systems with time-varying delay, and subsequently, Niu et al. [21] considered the related problem for Itô stochastic systems with Markovian switching. Further, Niu et al. paid some efforts to solving the switching problem between the sliding surface functions in [21]. However, it should be pointed out that the SMC design is under some restricted constraints (see (17) in [21]) which makes their results somewhat conservative. Thus, it is highly desirable to find a new SMC switching design to avoid those constraints. In this paper, it should be noted that the switching is arbitrary over an average dwell time, but not in the form of Markovian switching as proposed in [21,20]. To the authors' knowledge, there are few results reported on the SMC of arbitrary switching stochastic systems. In fact, investigating this research problem would be difficult due to the fact that the probability distribution of switching is not available. Many open questions still remain unsolved. We shall address some of these problems when considering the SMC of the switched stochastic systems:

- Q1. How to find the sliding surface function to avoid the repetitive jumps of the trajectories of the state components between sliding surfaces which may lead to possible instability? Also, how to avoid those restricted constraints in [21]?
- Q2. How to synthesize a SMC law so as to ensure the attraction of the sliding surface when the system changes from one mode to another under arbitrary switching?
- Q3. When some of the states are not available in a system, how to design an observer to estimate the states, and design observer-based SMC?

Motivated by the above questions, in this paper, we are interested in investigating the observer design and the SMC problems for a class of continuous-time switched stochastic system. The design objectives will be implemented as follows:

1. Synthesize a SMC law for the switched stochastic system when the system states are available for feedback.
2. Assume that some of the system state components are not available. In such a case, design a sliding mode observer first to estimate the unmeasured system state components, and then synthesize a SMC via the estimated system states.

The rest of this paper is organized as follows. The SMC problem of switched stochastic systems is formulated in Section 2. The main results of the SMC problem is presented in Section 3. The observer-based SMC synthesis is given in Section 4. Numerical examples are provided in Section 5 and we conclude this paper in Section 6.

Notations. The notations used throughout the paper are standard. The superscript "T" denotes matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent the identity matrix and a zero matrix, respectively; diag(...) stands for a block-diagonal matrix; $\lambda_{\min}(\cdot)$ ($\lambda_{\max}(\cdot)$) denotes the minimum (maximum) eigenvalue of a matrix. $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. For a vector $a = (a_i) \in \mathbb{R}^n$, $|a| \triangleq \sum_{i=1}^{n} |a_i|$ denotes the 1-norm of the vector a. $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space with Ω the sample space, $\mathcal{F}$ the σ-algebra of subsets of the sample space, and $\mathcal{P}$ the probability measure. $\mathbf{E}\{\cdot\}$ denotes the expectation operator with respect to probability measure $\mathcal{P}$. In symmetric block matrices or long matrix expressions, we use a star ($\star$) to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. System description and preliminaries

In the past decades, much attention has been focused on the study of Itô stochastic systems, since stochastic modeling has come to play an important role in many branches of science and industry. The extensive applications of Itô stochastic systems include population dynamics, macroeconomics, chemical reactor control, time-sharing and random round-off errors in computer operation, and other areas [22]. Readers can be referred to [23,24] for a detailed account of the Itô stochastic systems. In this paper, we will consider a class of switched Itô stochastic systems, which are established on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and described by Itô's form

$$(\Sigma): \mathrm{d}x(t) = [A(\beta)x(t) + B(u(t,\beta) + F(\beta)f(x,t))]\mathrm{d}t + D(\beta)x(t)\mathrm{d}\omega(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input; $\omega(t)$ is a one-dimensional $(1-D)$ Brownian motion satisfying $\mathbf{E}\{\mathrm{d}\omega(t)\} = 0$ and $\mathbf{E}\{\mathrm{d}\omega^2(t)\} = \mathrm{d}t$. Matrix B is assumed to be of full column rank. $f(x,t) \in \mathbb{R}^n$ is an unknown nonlinear function satisfying

$$\|F(\beta)f(x,t)\| \le \phi(\beta), \quad \beta \in \mathcal{N}, \tag{2}$$

where $\phi(\beta) > 0$ are scalars. In the system (Σ), $\{(A(\beta), D(\beta), F(\beta)) : \beta \in \mathcal{N}\}$ is a family of matrices parameterized by an index set $\mathcal{N} = \{1, 2, \ldots, N\}$ and $\beta : \mathbb{R} \to \mathcal{N}$ is a piecewise constant function of time t called a switching signal. At a given time t, the value of $\beta(t)$, denoted by β for simplicity, might depend on t or $x(t)$, or both, or may be generated by any other hybrid scheme. Therefore, the switched stochastic system effectively switches amongst N subsystems with the switching sequence controlled by $\beta(t)$. We assume that the value of $\beta(t)$ is unknown, but its instantaneous value is available in real time.

For each possible value $\beta(t) = i$, $i \in \mathcal{N}$, we will denote the system matrices associated with mode i by $A(i) = A(\beta)$, $D(i) = D(\beta)$, $F(i) = F(\beta)$, where $A(i)$, $D(i)$ and $F(i)$ are constant matrices. Corresponding to the switching signal β, we have the switching sequence $\{(i_0, t_0), (i_1, t_1), \ldots, (i_k, t_k), \ldots, |i_k \in \mathcal{N}, k = 0, 1, \ldots\}$ with $t_0 = 0$, which means that the i_kth subsystem is activated when $t \in [t_k, t_{k+1})$.

Assumption 1. Matrix B is of full column rank and the matrix pair $(A(i), B)$, $i \in \mathcal{N}$ is assumed to be controllable.

The autonomous system of (1) can be formulated as

$$(\Pi): \mathrm{d}x(t) = A(\beta)x(t)\mathrm{d}t + D(\beta)x(t)\mathrm{d}\omega(t). \tag{3}$$

Definition 1. The equilibrium $x^* = 0$ of system (3) is said to be mean-square exponentially stable under $\beta(t)$ if its solution $x(t)$ satisfies

$$\mathbf{E}\{\|x(t)\|^2\} < \eta\|x(t_0)\|^2 \mathrm{e}^{-\lambda(t-t_0)}, \quad \forall t \ge t_0, \text{ for } \eta \ge 1 \text{ and } \lambda > 0.$$

For the switching signal $\beta(t)$, we revisit the average dwell time property from the following definition.

Definition 2 ([25]). For any $T_2 > T_1 \geq 0$, let $N_\beta(T_1, T_2)$ denote the number of switchings of $\beta(t)$ over (T_1, T_2). If $N_\beta(T_1, T_2) \leq N_0 + (T_2 - T_1)/T_a$ holds for $T_a > 0$ and $N_0 \geq 0$. Then, T_a is called the average dwell time.

As commonly used in the literature, we choose $N_0 = 0$ in Definition 2.

For stochastic systems, Itô's formula plays an important role in the stability analysis. We cite the following result here.

Lemma 1 (*[24]. Itô's formula*). *Let $x(t)$ be an n-dimensional Itô's process on $t \geq 0$ with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}\omega(t),$$

where $f(t) \in \mathbb{R}^n$ and $g(t) \in \mathbb{R}^{n\times m}$. Let $V(x(t), t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$. Then, $V(x(t), t)$ is a real-valued Itô's process with its stochastic differential given by

$$\begin{aligned}\mathrm{d}V(x(t), t) &= \mathscr{L}V(x(t), t)\mathrm{d}t + V_x(x(t), t)g(t)\mathrm{d}\omega(t)\\ \mathscr{L}V(x(t), t) &= V_t(x(t), t) + V_x(x(t), t)f(t)\\ &\quad + \frac{1}{2}\mathrm{trace}(g^T(t)V_{xx}(x(t), t)g(t)),\end{aligned}$$

where $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ denotes the family of all real-valued functions $V(x(t), t)$ such that they are continuously twice differentiable in x and t. If $V(x(t), t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$, we set

$$\begin{aligned}V_t(x(t), t) &= \frac{\partial V(x(t), t)}{\partial t},\\ V_x(x(t), t) &= \left(\frac{\partial V(x(t), t)}{\partial x_1}, \ldots, \frac{\partial V(x(t), t)}{\partial x_n}\right),\\ V_{xx}(x(t), t) &= \left(\frac{\partial^2 V(x(t), t)}{\partial x_i x_j}\right)_{n\times n}.\end{aligned}$$

Lemma 2 (*[26]*). *For any real vectors $z, y \in \mathbb{R}^n$ and for any symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$,*

$$-2z^T y \leq z^T X^{-1} z + y^T X y.$$

To end this subsection, we recall the following lemma, which will play a key role in the derivation of our main results in this paper.

Lemma 3. *Given a scalar $\alpha > 0$, suppose there exist matrices $P(i) > 0$ such that for $i \in \mathscr{N}$,*

$$\begin{bmatrix} P(i)A(i) + A^T(i)P(i) + \alpha P(i) & D^T(i)P(i)\\ \star & -P(i)\end{bmatrix} < 0. \tag{4}$$

Then the switched stochastic system (Π) in (3) is mean-square exponentially stable for any switching signal with average dwell time satisfying $T_a > \frac{\ln\mu}{\alpha}$ with $\mu \geq 1$ and satisfying

$$P(i) \leq \mu P(j), \quad \forall i, j \in \mathscr{N}. \tag{5}$$

Moreover, an estimate of the mean-square of the state decay is given by

$$\mathbf{E}\{\|x(t)\|^2\} < \eta \mathrm{e}^{-\lambda t}\|x(0)\|^2, \tag{6}$$

where

$$\begin{aligned}\lambda &= \alpha - \frac{\ln\mu}{T_a} > 0, \qquad \eta = \frac{b}{a} > 1,\\ a &= \min_{\forall i\in\mathscr{N}} \lambda_{\min}(P(i)), \qquad b = \max_{\forall i\in\mathscr{N}} \lambda_{\max}(P(i)).\end{aligned} \tag{7}$$

The desired result can be obtained by applying the piecewise Lyapunov function approach together with the average dwell time technique, the detailed proof can be referred to [27].

3. Sliding mode control

3.1. Sliding mode dynamics analysis

In this subsection, we will first analyze the sliding mode dynamics. Since B is of full column rank by assumption, there exists a nonsingular matrix $\mathscr{T}$ such that

$$\mathscr{T}B = \begin{bmatrix} 0_{(n-m)\times m}\\ B_1\end{bmatrix}, \tag{8}$$

where $B_1 \in \mathbb{R}^{m\times m}$ is nonsingular. Taking a singular value decomposition of B, we have

$$B = U\begin{bmatrix} 0_{(n-m)\times m}\\ \Gamma\end{bmatrix}W^T, \tag{9}$$

where $U \triangleq \begin{bmatrix} U_1 & U_2\end{bmatrix}$ and $W \in \mathbb{R}^{m\times m}$ are unitary matrices with $U_1 \in \mathbb{R}^{n\times(n-m)}$, $U_2 \in \mathbb{R}^{n\times m}$; $\Gamma \in \mathbb{R}^{m\times m}$ is a diagonal positive-definite matrix. For convenience, choose $\mathscr{T} = U^T$. Then, by the transformation $z(t) = \mathscr{T}x(t)$, system (1) becomes

$$\begin{aligned}\mathrm{d}z(t) &= \{\mathscr{T}A(i)\mathscr{T}^{-1}z(t) + \mathscr{T}B[u(t, i) + F(i)f(x, t)]\}\mathrm{d}t\\ &\quad + \mathscr{T}D(i)\mathscr{T}^{-1}z(t)\mathrm{d}\omega(t).\end{aligned} \tag{10}$$

Let $\bar{A}(i) \triangleq \mathscr{T}A(i)\mathscr{T}^{-1}$, $\bar{B} \triangleq \mathscr{T}B$, $\bar{D}(i) \triangleq \mathscr{T}D(i)\mathscr{T}^{-1}$ and

$$\begin{aligned}\bar{A}(i) &\triangleq \begin{bmatrix}\bar{A}_{11}(i) & \bar{A}_{12}(i)\\ \bar{A}_{21}(i) & \bar{A}_{22}(i)\end{bmatrix},\\ \bar{D}(i) &\triangleq \begin{bmatrix}\bar{D}_{11}(i) & \bar{D}_{12}(i)\\ \bar{D}_{21}(i) & \bar{D}_{22}(i)\end{bmatrix}, \qquad \bar{B} \triangleq \begin{bmatrix} 0_{(n-m)\times m}\\ B_1\end{bmatrix}.\end{aligned} \tag{11}$$

Then, let $z(t) \triangleq \begin{bmatrix} z_1^T(t) & z_2^T(t)\end{bmatrix}^T$ with $z_1(t) \in \mathbb{R}^{n-m}$ and $z_2(t) \in \mathbb{R}^m$. Thus, (10) can be written as

$$\begin{aligned}\begin{bmatrix}\mathrm{d}z_1(t)\\ \mathrm{d}z_2(t)\end{bmatrix} &= \left\{\bar{A}(i)\begin{bmatrix} z_1(t)\\ z_2(t)\end{bmatrix} + \bar{B}(u(t, i) + F(i)f(x, t))\right\}\mathrm{d}t\\ &\quad + \bar{D}(i)\begin{bmatrix} z_1(t)\\ z_2(t)\end{bmatrix}\mathrm{d}\omega(t).\end{aligned} \tag{12}$$

According to the SMC theory [28], we know that the first subsystem in (12) represents the sliding mode dynamics. Choose the following linear sliding surface function

$$s(t) = Kz_1(t) + z_2(t), \tag{13}$$

where $K \in \mathbb{R}^{m\times(n-m)}$ is the parameter to be found within the autonomous part of the above system.

Remark 1. It should be pointed out that the sliding surface function defined in (13) does not switch with the switching signal β (we use K instead of $K(\beta)$ in (13), which means that K is independent of β), that is, there is a unique (non-switched) sliding surface. While, in [20] the parameter-dependent sliding surface function was used. The reason why we chose the parameter-independent sliding surface function is to avoid repetitive jumps of the trajectories of the state components of the closed-loop system between sliding surfaces which may cause instability. This has given a partial answer to Q1 above.

Remark 2. Since a decomposition is performed according to (8)–(9), the SMC in this paper is then synthesized based on the transformed system (12). Moreover, the methods used in this paper are based on the linear sliding surface function (13) while the integral sliding surface function is used in [21]. Hence, those restricted constraints in [21] are no longer imposed in the setting of this SMC synthesis. This has given the remaining answer to Q1 above. On the other hand, comparing the linear sliding surface function in (13) of our work with the integral sliding surface function in [21], it is shown obviously that the linear sliding surface function is simple thus easier for implementation in practice.

When the system trajectories reach onto the sliding surface $s(t) = 0$, that is, $z_2(t) = -Kz_1(t)$, the sliding mode dynamics is attained. Substituting $z_2(t) = -Kz_1(t)$ into the first subsystem in (12) yields the sliding mode dynamics:

$$\begin{aligned} dz_1(t) = {} & [(\bar{A}_{11}(i) - \bar{A}_{12}(i)K)z_1(t)]dt \\ & + [(\bar{D}_{11}(i) - \bar{D}_{12}(i)K)z_1(t)]d\omega(t). \end{aligned} \tag{14}$$

We will analyze the stability of the sliding mode dynamics in (14) based on Lemma 3.

Theorem 1. *For a given constant $\alpha > 0$, suppose there exist matrices $\mathscr{F} > 0$, $\mathscr{P}(i) > 0$, $\mathscr{Q}(i) > 0$, $\mathcal{F} > 0$, $\mathcal{P}(i) > 0$, $\mathcal{Q}(i) > 0$ and $\mathcal{K}$ such that for $i \in \mathscr{N}$,*

$$\begin{bmatrix} \tilde{\Upsilon}_{11}(i) & \tilde{\Upsilon}_{12}(i) & (\bar{D}_{11}(i)\mathcal{F} - \bar{D}_{12}(i)\mathcal{K})^T \\ \star & -2\mathcal{F} & 0 \\ \star & \star & -\mathscr{Q}(i) \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} -\mathcal{P}(i) & \mathscr{F} \\ \star & -\mathcal{Q}(i) \end{bmatrix} \le 0, \tag{16}$$

$$\mathscr{F}\mathcal{F} = I, \qquad \mathscr{P}(i)\mathcal{P}(i) = I, \qquad \mathscr{Q}(i)\mathcal{Q}(i) = I, \tag{17}$$

where

$$\tilde{\Upsilon}_{11}(i) \triangleq \bar{A}_{11}(i)\mathcal{F} + \mathcal{F}\bar{A}_{11}^T(i) - \bar{A}_{12}(i)\mathcal{K} - \mathcal{K}^T\bar{A}_{12}^T(i) + \alpha\mathscr{P}(i)$$
$$\tilde{\Upsilon}_{12}(i) \triangleq \mathscr{P}(i) - \mathcal{F} + (\bar{A}_{11}(i)\mathcal{F} - \bar{A}_{12}(i)\mathcal{K})^T.$$

Then the sliding mode dynamics in (14) is mean-square exponentially stable for any switching signal with average dwell time satisfying $T_a > \frac{\ln \mu}{\alpha}$ with $\mu \ge 1$ and satisfying

$$\mathscr{P}(i) \le \mu\mathscr{P}(j), \qquad \mathcal{P}(i) \le \mu\mathcal{P}(j), \qquad \forall i, j \in \mathscr{N}. \tag{18}$$

Moreover, if the conditions above are feasible, the matrix K in (13) is given by $K = \mathcal{K}\mathcal{F}^{-1} = \mathcal{K}\mathscr{F}$, thus, the sliding surface function can be described by

$$s(t) = \mathcal{K}\mathcal{F}^{-1}z_1(t) + z_2(t) = \mathcal{K}\mathscr{F}z_1(t) + z_2(t) = 0. \tag{19}$$

Proof. According to Lemma 3 and introducing a slack matrix $\mathscr{F}$, it is not difficult to see that the switched stochastic system (Π) in (3) is mean-square exponentially stable if there exist matrices $P(i) > 0$ and $\mathscr{F} > 0$ such that the following condition holds:

$$\begin{bmatrix} \mathscr{F}A(i) + A^T(i)\mathscr{F} + \alpha P(i) & P(i) - \mathscr{F} + A^T(i)\mathscr{F} & D^T(i) \\ \star & -2\mathscr{F} & 0 \\ \star & \star & -P^{-1}(i) \end{bmatrix} < 0. \tag{20}$$

By performing a projection transformation to (20) by

$$\Lambda(i) \triangleq \begin{bmatrix} I & 0 \\ A(i) & 0 \\ 0 & P(i) \end{bmatrix},$$

we can see that the condition in (20) implies (4) in Lemma 3. By the above analysis, we know that if there exist matrices $P(i) > 0$ and a slack matrix $\mathscr{F} > 0$ such that for $i \in \mathscr{N}$,

$$\begin{bmatrix} \Upsilon_{11}(i) & \Upsilon_{12}(i) & (\bar{D}_{11}(i) - \bar{D}_{12}(i)K)^T \\ \star & -2\mathscr{F} & 0 \\ \star & \star & -P^{-1}(i) \end{bmatrix} < 0, \tag{21}$$

where

$$\Upsilon_{11}(i) \triangleq \mathscr{F}(\bar{A}_{11}(i) - \bar{A}_{12}(i)K) + (\bar{A}_{11}(i) - \bar{A}_{12}(i)K)^T\mathscr{F} + \alpha P(i),$$
$$\Upsilon_{12}(i) \triangleq P(i) - \mathscr{F} + (\bar{A}_{11}(i) - \bar{A}_{12}(i)K)^T\mathscr{F},$$

then the sliding mode dynamics in (14) is mean-square exponentially stable. Defining $\mathcal{F} \triangleq \mathscr{F}^{-1}$, $\mathscr{P}(i) \triangleq \mathcal{F}P(i)\mathcal{F}$ and performing a congruence transformation to (21) with $\mathrm{diag}(\mathcal{F}, \mathcal{F}, I)$ gives

$$\begin{bmatrix} \bar{\Upsilon}_{11}(i) & \bar{\Upsilon}_{12}(i) & (\bar{D}_{11}(i)\mathcal{F} - \bar{D}_{12}(i)K\mathcal{F})^T \\ \star & -2\mathcal{F} & 0 \\ \star & \star & -\mathcal{F}\mathscr{P}^{-1}(i)\mathcal{F} \end{bmatrix} < 0, \tag{22}$$

where

$$\bar{\Upsilon}_{11}(i) \triangleq \bar{A}_{11}(i)\mathcal{F} + \mathcal{F}\bar{A}_{11}^T(i) - \bar{A}_{12}(i)K\mathcal{F} - \mathcal{F}K^T\bar{A}_{12}^T(i) + \alpha\mathscr{P}(i)$$
$$\bar{\Upsilon}_{12}(i) \triangleq \mathscr{P}(i) - \mathcal{F} + \left(\bar{A}_{11}(i)\mathcal{F} - \bar{A}_{12}(i)K\mathcal{F}\right)^T.$$

It can be seen that (22) holds if the following conditions hold

$$\begin{bmatrix} \bar{\Upsilon}_{11}(i) & \bar{\Upsilon}_{12}(i) & (\bar{D}_{11}(i)\mathcal{F} - \bar{D}_{12}(i)K\mathcal{F})^T \\ \star & -2\mathcal{F} & 0 \\ \star & \star & -\mathscr{Q}(i) \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} -\mathscr{P}^{-1}(i) & \mathcal{F}^{-1} \\ \star & -\mathscr{Q}^{-1}(i) \end{bmatrix} \le 0. \tag{24}$$

Let $\mathcal{K} \triangleq K\mathcal{F}$, $\mathcal{P}(i) \triangleq \mathscr{P}^{-1}(i)$ and $\mathcal{Q}(i) \triangleq \mathscr{Q}^{-1}(i)$, and we have (15) and (16). Moreover, considering (5) and noting $\mathscr{P}(i) \triangleq \mathcal{F}P(i)\mathcal{F}$ and $\mathcal{P}(i) \triangleq \mathscr{P}^{-1}(i)$, these yield (18). This completes the proof. □

Remark 3. It should be pointed out that we use the condition in (20), not (4), to solve the sliding surface function in Theorem 1. Notice that the matrix variables $P(i)$ in (4) are dependent on the switching set, while the newly introduced matrix variable $\mathscr{F}$ in (20) is fixed and it does not depend on the switching set. Since the sliding surface function in (13) is a parameter-independent function, the parameter $K = \mathcal{K}\mathcal{F}^{-1} = \mathcal{K}\mathscr{F}$ in (13) is guaranteed to be fixed given that the matrix variable $\mathscr{F}$ is fixed.

Remark 4. Notice that the condition in Theorem 1 is not a convex set due to the matrix equality constraints in (17). Several approaches have been proposed to solve such a nonconvex feasibility problem, among which the cone complementarity linearization (CCL) method [29] is the most commonly used one.

Problem SMDA (Sliding Mode Dynamics Analysis):

$$\min \qquad \mathrm{trace}\left(\mathscr{F}\mathcal{F} + \sum_{i\in\mathscr{N}}\mathscr{P}(i)\mathcal{P}(i) + \sum_{i\in\mathscr{I}}\mathscr{Q}(i)\mathcal{Q}(i)\right)$$

subject to (15)–(16), (18) and for $i \in \mathscr{N}$,

$$\begin{bmatrix} \mathscr{F} & I \\ I & \mathcal{F} \end{bmatrix} \ge 0, \qquad \begin{bmatrix} \mathscr{P}(i) & I \\ I & \mathcal{P}(i) \end{bmatrix} \ge 0, \qquad \begin{bmatrix} \mathscr{Q}(i) & I \\ I & \mathcal{Q}(i) \end{bmatrix} \ge 0. \tag{25}$$

Now using a cone complementarity method [29], we suggest the following nonlinear minimization problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 1.

Algorithm SMDA

Step 1. Find a feasible set $(\mathscr{F}^{(0)}, \mathcal{F}^{(0)}, \mathscr{P}^{(0)}(i), \mathcal{P}^{(0)}(i), \mathscr{Q}^{(0)}(i), \mathcal{Q}^{(0)}(i), \mathcal{K}^{(0)})$ satisfying (15)–(16), (18) and (25). Set $\kappa = 0$.

Step 2. Solve the following optimization problem

$$\min\ \text{trace}\begin{pmatrix} \mathscr{F}^{(\kappa)}\mathcal{F}+\mathscr{F}\mathcal{F}^{(\kappa)} \\ +\sum_{i\in\mathcal{N}}(\mathscr{P}^{(\kappa)}(i)\mathcal{P}(i)+\mathscr{P}(i)\mathcal{P}^{(\kappa)}(i) \\ +\mathscr{Q}^{(\kappa)}(i)\mathcal{Q}(i)+\mathscr{Q}(i)\mathcal{Q}^{(\kappa)}(i)) \end{pmatrix}$$

subject to (15)–(16), (18) and (25)

and denote f^* to be the optimized value.

Step 3. Substitute the obtained $(\mathscr{F},\mathcal{F},\mathscr{P}(i),\mathcal{P}(i),\mathscr{Q}(i),\mathcal{Q}(i),\mathcal{K})$ into (24). If (24) is satisfied, with

$$|f^*-(2+4N)(n-m)|<\sigma$$

for a sufficiently small scalar $\sigma>0$, then output $(\mathscr{F},\mathcal{F},\mathscr{P}(i),\mathcal{P}(i),\mathscr{Q}(i),\mathcal{Q}(i),\mathcal{K})$. EXIT.

Step 4. If $\kappa>\mathbb{N}$ where $\mathbb{N}$ is the maximum number of iterations allowed, EXIT.

Step 5. Set $\kappa=\kappa+1$, $(\mathscr{F}^{(\kappa)},\mathcal{F}^{(\kappa)},\mathscr{P}^{(\kappa)}(i),\mathcal{P}^{(\kappa)}(i),\mathscr{Q}^{(\kappa)}(i),\mathcal{Q}^{(\kappa)}(i),\mathcal{K}^{(\kappa)})=(\mathscr{F},\mathcal{F},\mathscr{P}(i),\mathcal{P}(i),\mathscr{Q}(i),\mathcal{Q}(i),\mathcal{K})$, and go to Step 2.

3.2. *Sliding mode controller synthesis*

To give an answer to Q2 above, in this subsection, we are in a position to synthesize a SMC law to drive the system trajectories onto the pre-defined sliding surface $s(t)=0$ in (19). The result is shown below:

Theorem 2. *Suppose (15) – (17) and (18) have solutions $\mathscr{F}>0,\mathcal{F}>0,\mathscr{P}(i)>0,\mathcal{P}(i)>0,\mathscr{Q}(i)>0,\mathcal{Q}(i)>0,\mathcal{K}$ and the linear sliding surface is given by (19). Then, the trajectory of the closed-loop system (12) can be driven onto the sliding surface $s(t)=0$ in a finite time with the control*

$$u(t,i)=-\gamma B_1^{-1}s(t)-(\phi(i)+\delta(i))\text{sign}(B_1^Ts(t)),\quad i\in\mathcal{N},\tag{26}$$

where $\delta(i)>0$, $i\in\mathcal{N}$ are constants and

$$\gamma\triangleq\frac{1}{2}\sup_{i\in\mathcal{N}}\lambda_{\max}[\mathcal{H}\bar{A}(i)\mathcal{H}^++(\mathcal{H}\bar{A}(i)\mathcal{H}^+)^T+(\mathcal{H}\bar{D}(i)\mathcal{H}^+)^T\mathcal{H}\bar{D}(i)\mathcal{H}^+],\tag{27}$$

where $\mathcal{H}\triangleq\begin{bmatrix}\mathcal{K}\mathcal{F}^{-1} & I\end{bmatrix}$ and $\mathcal{H}^+$ denotes the Moore–Penrose inverse of matrix $\mathcal{H}$.

Proof. We will prove that the control law (26) can drive the system trajectory onto the sliding surface $s(t)=0$. Consider the following switching function:

$$s(t)=\mathcal{K}\mathcal{F}^{-1}z_1(t)+z_2(t)\triangleq\mathcal{H}z(t),\tag{28}$$

where $\mathcal{H}$ is defined in (27). According to (10), rewritten as

$$\begin{aligned}dz(t)&=\{\bar{A}(i)z(t)+\bar{B}[u(t,i)+F(i)f(x,t)]\}dt\\&\quad+\bar{D}(i)z(t)d\omega(t),\end{aligned}\tag{29}$$

we have

$$\begin{aligned}ds(t)&=\mathcal{H}dz(t)=\{\mathcal{H}\bar{A}(i)z(t)\\&\quad+\mathcal{H}\bar{B}[u(t,i)+F(i)f(x,t)]\}dt+\mathcal{H}\bar{D}(i)z(t)d\omega(t)\\&=\{\mathcal{H}\bar{A}(i)\mathcal{H}^+s(t)+\mathcal{H}\bar{B}[u(t,i)+F(i)f(x,t)]\}dt\\&\quad+\mathcal{H}\bar{D}(i)\mathcal{H}^+s(t)d\omega(t),\end{aligned}\tag{30}$$

where $\mathcal{H}^+$ denotes the Moore–Penrose inverse of matrix $\mathcal{H}$.

Choose the following Lyapunov function:

$$W(t)\triangleq\frac{1}{2}s^T(t)s(t),\tag{31}$$

thus, along the solution of system (30), by Itô's formula, we obtain the stochastic differential as

$$dW(t)=\mathscr{L}W(t)dt+s^T(t)\mathcal{H}\bar{D}(i)\mathcal{H}^+s(t)d\omega(t),\tag{32}$$

where

$$\begin{aligned}\mathscr{L}W(t)&=s^T(t)\{\mathcal{H}\bar{A}(i)\mathcal{H}^+s(t)+\mathcal{H}\bar{B}[u(t,i)+F(i)f(x,t)]\}\\&\quad+\frac{1}{2}s^T(t)(\mathcal{H}\bar{D}(i)\mathcal{H}^+)^T\mathcal{H}\bar{D}(i)\mathcal{H}^+s(t)\\&=s^T(t)\{\mathcal{H}\bar{A}(i)\mathcal{H}^+s(t)+B_1[u(t,i)+F(i)f(x,t)]\}\\&\quad+\frac{1}{2}s^T(t)(\mathcal{H}\bar{D}(i)\mathcal{H}^+)^T\mathcal{H}\bar{D}(i)\mathcal{H}^+s(t)\\&=\frac{1}{2}s^T(t)[\mathcal{H}\bar{A}(i)\mathcal{H}^++(\mathcal{H}\bar{A}(i)\mathcal{H}^+)^T\\&\quad+(\mathcal{H}\bar{D}(i)\mathcal{H}^+)^T\mathcal{H}\bar{D}(i)\mathcal{H}^+]s(t)\\&\quad+s^T(t)B_1[u(t,i)+F(i)f(x,t)].\end{aligned}\tag{33}$$

Substituting the SMC law (26) into (33) and noting that $\|s^T(t)B_1\|\le|s^T(t)B_1|$, we have

$$\begin{aligned}\mathscr{L}W(t)&=\frac{1}{2}s^T(t)[\mathcal{H}\bar{A}(i)\mathcal{H}^++(\mathcal{H}\bar{A}(i)\mathcal{H}^+)^T\\&\quad+(\mathcal{H}\bar{D}(i)\mathcal{H}^+)^T\mathcal{H}\bar{D}(i)\mathcal{H}^+]s(t)-\gamma s^T(t)s(t)-(\phi(i)\\&\quad+\delta(i))|s^T(t)B_1|+s^T(t)B_1F(i)f(x,t)\\&\le-\delta(i)\|s^T(t)B_1\|-\phi(i)\|s^T(t)B_1\|+s^T(t)B_1F(i)f(x,t)\\&\le-\delta(i)\|s^T(t)B_1\|\\&\le-\delta(i)\sqrt{\lambda_{\min}(B_1B_1^T)}\|s(t)\|\\&=-\delta(i)\sqrt{\lambda_{\min}(B_1B_1^T)}(s^T(t)s(t))^{1/2}\\&\le-\sqrt{2\lambda_{\min}(B_1B_1^T)}\min_{\forall i\in\mathcal{N}}(\delta(i))W^{1/2}(t)\\&\triangleq-\delta W^{1/2}(t)<0\end{aligned}\tag{34}$$

where $\delta\triangleq\sqrt{2\lambda_{\min}(B_1B_1^T)}\min_{\forall i\in\mathcal{N}}(\delta(i)c)>0$.

For an arbitrary piecewise constant switching signal β, and for any $t>0$, we let $0=t_0<t_1<\cdots<t_k<\cdots$, $k=0,1,\ldots$, denote the switching points of β over the interval $(0,t)$. As mentioned earlier, the i_kth subsystem is activated when $t\in[t_k,t_{k+1})$. Integrating $\mathscr{L}W(t)\le-\delta W^{1/2}(t)$ from t_k to t and t_{k-1} to t_k, $k=1,2,\ldots$, and taking expectations, we have

$$\mathbf{E}(W^{1/2}(t))-\mathbf{E}(W^{1/2}(t_k))\le-\frac{1}{2}\delta(t-t_k)$$

$$\mathbf{E}(W^{1/2}(t_k))-\mathbf{E}(W^{1/2}(t_{k-1}))\le-\frac{1}{2}\delta(t_k-t_{k-1})$$

$$\vdots$$

$$\mathbf{E}(W^{1/2}(t_1))-\mathbf{E}(W^{1/2}(0))\le-\frac{1}{2}\delta(t_1-0).$$

Summing the terms on both sides of the above inequalities gives

$$\mathbf{E}(W^{1/2}(t))-\mathbf{E}(W^{1/2}(0))\le-\frac{1}{2}\delta t.$$

Thus, it can seen that there exists a time $t^*\le2\mathbf{E}(W^{1/2}(0))/\delta$ such that $W(t)=0$, and consequently $s(t)=0$, for $t\ge t^*$, which means that the system trajectories can reach onto the predefined sliding surface in a finite time. This completes the proof. □

4. Observer-based sliding mode control

In this section, we shall study the SMC problem under the assumption that some of the system state components are not available. We will first utilize a state observer to generate the estimate of unmeasured states, and then synthesize a SMC law based on the state estimates. To begin this, we give the following measured output for the switched stochastic system (Σ):

$$y(t) = C(i)x(t), \quad i \in \mathcal{N}, \tag{35}$$

where $y(t) \in \mathbb{R}^p$ is the measured output. We design the following sliding mode observer to estimate the states of the switched stochastic system (Σ):

$$\begin{aligned} \mathrm{d}\hat{x}(t) = {} & \{A(i)\hat{x}(t) + B[u(t,i) + v(t,i)] \\ & + L(i)[y(t) - C(i)\hat{x}(t)]\}\mathrm{d}t, \end{aligned} \tag{36}$$

where $\hat{x}(t) \in \mathbb{R}^n$ represents the estimate of the system state $x(t)$, $L(i) \in \mathbb{R}^{n\times p}$, $i \in \mathcal{N}$ are the observer gains to be designed later, and the control term $v(t,i)$ is chosen to eliminate the effect of nonlinear function $F(i)f(x,t)$.

Let $e(t) \triangleq x(t) - \hat{x}(t)$ denote the estimation error. According to (1), (35) and (36), the estimation error dynamics is obtained as

$$\begin{aligned} \mathrm{d}e(t) = {} & \{[A(i) - L(i)C(i)]e(t) - B[v(t,i) - F(i)f(x,t)]\}\mathrm{d}t \\ & + D(i)x(t)\mathrm{d}\omega(t), \end{aligned} \tag{37}$$

which can be reformulated as

$$\begin{aligned} \mathrm{d}e(t) = {} & \{[A(i) - L(i)C(i)]e(t) - B[v(t,i) - F(i)f(x,t)]\}\mathrm{d}t \\ & + D(i)e(t)\mathrm{d}\omega(t) + D(i)\hat{x}(t)\mathrm{d}\omega(t). \end{aligned} \tag{38}$$

Remark 5. Notice from (38) that the estimation error dynamics corresponds to a switched stochastic system, and is dependent on the observer feedback matrix $L(i)$ and state estimates $\hat{x}(t)$. This means that the stability analysis of the error dynamics (38) is not independent of the observer dynamics (36).

Define the following sliding surface functions in the state estimation space and in the state-estimation error space, respectively, namely:

$$s_x(t) = B^T X\left[\hat{x}(t) + \int_0^t BG\hat{x}(\sigma)\mathrm{d}\sigma\right], \tag{39}$$

$$s_e(t) = B^T X e(t), \tag{40}$$

where $X \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and $G \in \mathbb{R}^{m\times n}$ is a constant matrix which is chosen such that the matrices $(A(i) - BG)$, $i \in \mathcal{N}$ are Hurwitz.

The state estimate-based SMC laws are designed as

$$u(t,i) = -(\pi(i) + \chi(t) + \kappa(i) + \phi(i))\mathrm{sign}(s_x(t)), \tag{41}$$

$$v(t,i) = (\kappa(i) + \phi(i))\mathrm{sign}(s_e(t)), \tag{42}$$

where $\pi(i) > 0$, $\kappa(i) > 0$, $i \in \mathcal{N}$ are small constants, and the switching gain $\chi(t)$ is given as

$$\begin{aligned} \chi(t) = {} & \max_{i\in\mathcal{N}} \|(B^TXB)^{-1}\|\{[\|B^TX(A(i)+BG)\| \\ & + \|B^TXL(i)C(i)\|]\|\hat{x}(t)\| + \|B^TXL(i)\|\,\|y(t)\|\}. \end{aligned}$$

We will show in the following theorem that the sliding motion will be driven onto the specified sliding surface $s_x(t) = 0$ in a finite time and be maintained there subsequently.

Theorem 3. *The trajectories of the systems (36) and (38) can be driven onto the sliding surface $s_x(t) = 0$ in a finite time by the observer-based SMC (41)–(42).*

Proof. Select the following Lyapunov function:

$$V(t) = \frac{1}{2}s_x^T(t)(B^TXB)^{-1}s_x(t). \tag{43}$$

Noting $\|s_x(t)\| \le |s_x(t)|$ and $s_x^T(t)\mathrm{sign}(s_e(t)) \le |s_x(t)|$, thus we have

$$\begin{aligned} \dot{V}(t) &= s_x^T(t)(B^TXB)^{-1}\dot{s}_x(t) \\ &= s_x^T(t)(B^TXB)^{-1}B^TX(\dot{\hat{x}}(t) + BG\hat{x}(t)) \\ &= s_x^T(t)(B^TXB)^{-1}B^TX\{(A(i)+BG)\hat{x}(t) \\ &\quad + B(u(t,i)+v(t,i)) + L(i)(y(t) - C(i)\hat{x}(t))\} \\ &\le \|s_x^T(t)\|\,\|(B^TXB)^{-1}\|\{\|B^TX(A(i)+BG)\hat{x}(t)\| \\ &\quad + \|B^TXL(i)y(t)\| + \|B^TXL(i)C(i)\hat{x}(t)\|\} \\ &\quad - (\pi(i) + \chi(t) + \kappa(i) + \phi(i))|s_x(t)| + (\kappa(i) + \phi(i))|s_x(t)| \\ &\le \|s_x^T(t)\|\,\|(B^TXB)^{-1}\|\{\|B^TX(A(i)+BG)\hat{x}(t)\| \\ &\quad + \|B^TXL(i)y(t)\| + \|B^TXL(i)C(i)\hat{x}(t)\|\} \\ &\quad - \pi(i)\|s_x(t)\| - \chi(t)\|s_x(t)\| \\ &\le -\pi(i)\|s_x(t)\| \le -\vartheta V^{\frac{1}{2}}(t), \end{aligned} \tag{44}$$

where $\vartheta \triangleq \sqrt{\frac{2}{\lambda_{\min}(B^TXB)}}\min_{i\in\mathcal{N}}(\pi(i)) > 0$.

Therefore, by using the same methods as in the proof Theorem 2, we can conclude that the state trajectories of the observer dynamics (36) can be driven onto the sliding surface $s_x(t) = 0$ by the observer-based SMC (41)–(42) in a finite time. This completes the proof. □

According to the sliding mode theory [28], it follows from $\dot{s}_x(t) = 0$ that the following equivalent control law can be obtained:

$$\begin{aligned} u_{\mathrm{eq}}(t,i) = {} & -(B^TXB)^{-1}B^TX\{[A(i)+BG]\hat{x}(t) \\ & + L(i)[y(t) - C(i)\hat{x}(t)]\}. \end{aligned} \tag{45}$$

Substituting (45) into (36) yields the sliding mode dynamics in the state estimation space, which can be formulated as

$$\begin{aligned} \mathrm{d}\hat{x}(t) = {} & \{[I - B(B^TXB)^{-1}B^TX][A(i)\hat{x}(t) \\ & + L(i)C(i)e(t)] - BG\hat{x}(t)\}\mathrm{d}t. \end{aligned} \tag{46}$$

Remark 6. The above work is to design a SMC law based on the estimated system states, such that the system state trajectories can be driven onto the predefined sliding surface in a finite time. Notice from Theorem 3 that the sliding motion in the state estimation space is attained. This means that the dynamics of the overall closed-loop systems (36) and (38) will reduce to a system composed of the estimation error dynamics (38) and the sliding mode dynamics in the state estimation space (46).

In the following theorem, the sufficient condition for the stability is given in terms of LMIs for the overall closed-loop system composed of the estimation error dynamics (38) and the sliding mode dynamics in the state estimation space (46).

Theorem 4. *Consider the switched stochastic system (Σ) in (1) with (35). Its unmeasured states are estimated by the observer (36). The sliding surface functions in the state estimation space and in the state-estimation error space are chosen as (39)–(40), and the observer-based SMC law is synthesized by (41)–(42). If there exist matrices $X > 0$ and $\mathcal{L}(i)$ such that for $i \in \mathcal{N}$,*

$$\begin{bmatrix} \Pi_{11}(i) & \Pi_{12}(i) & \sqrt{2}XB & 0 \\ \star & \Pi_{22}(i) & 0 & C^T(i)\mathcal{L}^T(i) \\ \star & \star & -B^TXB & 0 \\ \star & \star & \star & -X \end{bmatrix} < 0, \tag{47}$$

where

$$\begin{aligned} \Pi_{11}(i) \triangleq {} & XA(i) + A^T(i)X - XBG - G^TB^TX \\ & + A^T(i)XA(i) + D^T(i)XD(i), \end{aligned}$$

$$\Pi_{12}(i) \triangleq \mathcal{L}(i)C(i) + D^T(i)XD(i),$$

$$\begin{aligned} \Pi_{22}(i) \triangleq {} & XA(i) + A^T(i)X - \mathcal{L}(i)C(i) \\ & - C^T(i)\mathcal{L}^T(i) + D^T(i)XD(i), \end{aligned}$$

then the overall closed-loop switched stochastic system is stable. Moreover, the observer gain is given by

$$L(i) = X^{-1}\mathcal{L}(i), \quad i \in \mathcal{N}. \tag{48}$$

Proof. Select the following Lyapunov function:

$$\tilde{V}(\hat{x}, e, i) \triangleq \tilde{V}(\hat{x}, i) + \tilde{V}(e, i),$$
$$\tilde{V}(\hat{x}, i) \triangleq \frac{1}{2}\hat{x}^T(t)X\hat{x}(t),$$
$$\tilde{V}(e, i) \triangleq \frac{1}{2}e^T(t)Xe(t). \tag{49}$$

Thus, along the solution of systems (38) and (46), by Lemma 1, we have

$$\mathcal{L}\tilde{V}(\hat{x}, i) = \hat{x}^T(t)X\{[I - B(B^TXB)^{-1}B^TX] \times [A(i)\hat{x}(t) + L(i)C(i)e(t)] - BG\hat{x}(t)\} \tag{50}$$

$$\mathcal{L}\tilde{V}(e, i) = e^T(t)X\{[A(i) - L(i)C(i)]e(t) - B[v(t, i) - F(i)f(x, t)]\} + \frac{1}{2}x^T(t)D^T(i)XD(i)x(t). \tag{51}$$

Thus, we have

$$\begin{aligned}\mathcal{L}\tilde{V}(\hat{x}, e, i) &= \frac{1}{2}\hat{x}^T(t)(XA(i) + A^T(i)X)\hat{x}(t) + \hat{x}^T(t)XL(i)C(i)e(t)\\ &\quad - \hat{x}^T(t)XB(B^TXB)^{-1}B^TXA(i)\hat{x}(t)\\ &\quad - \hat{x}^T(t)XB(B^TXB)^{-1}B^TXL(i)C(i)e(t)\\ &\quad - \hat{x}^T(t)XBG\hat{x}(t) + e^T(t)X[A(i) - L(i)C(i)]e(t)\\ &\quad - e^T(t)XB[v(t, i) - F(i)f(x, t)]\\ &\quad + \frac{1}{2}x^T(t)D^T(i)XD(i)x(t).\end{aligned} \tag{52}$$

Notice (42) and $\|s_e(t)\| \le |s_e(t)|$. Thus,

$$\begin{aligned}&- e^T(t)XB[v(t, i) - F(i)f(x, t)]\\ &= -s_e^T(t)(\kappa(i) + \phi(i))\text{sign}(s_e(t)) + s_e^T(t)F(i)f(x, t)\\ &\le -(\kappa(i) + \phi(i))|s_e^T(t)| + \phi(i)\|s_e^T(t)\|\\ &\le -\kappa(i)\|s_e^T(t)\| < 0.\end{aligned} \tag{53}$$

On the other hand, by Lemma 2, we have

$$-\hat{x}^T(t)XB(B^TXB)^{-1}B^TXA(i)\hat{x}(t) \le \frac{1}{2}\{\hat{x}^T(t)XB(B^TXB)^{-1}B^TX\hat{x}(t) + \hat{x}^T(t)A^T(i)XA(i)\hat{x}(t)\},$$

and

$$-\hat{x}^T(t)XB(B^TXB)^{-1}B^TXL(i)C(i)e(t) \le \frac{1}{2}\{\hat{x}^T(t)XB(B^TXB)^{-1}B^TX\hat{x}(t) + e^T(t)C^T(i)L^T(i)XL(i)C(i)e(t)\}.$$

Considering (50)–(53), we have

$$\mathcal{L}\tilde{V}(\hat{x}, e, i) \le \frac{1}{2}\zeta^T(t)\Omega(i)\zeta(t), \tag{54}$$

where $\zeta(t) \triangleq \begin{bmatrix}\hat{x}(t)\\ e(t)\end{bmatrix}$ and $\Omega(i) \triangleq \begin{bmatrix}\Omega_{11}(i) & \Omega_{12}(i)\\ \star & \Omega_{22}(i)\end{bmatrix}$ with

$$\Omega_{11}(i) \triangleq X(A(i) - BG) + (A(i) - BG)^TX + 2XB(B^TXB)^{-1} \times B^TX + A^T(i)XA(i) + D^T(i)XD(i),$$
$$\Omega_{12}(i) \triangleq XL(i)C(i) + D^T(i)XD(i),$$
$$\Omega_{22}(i) \triangleq XA(i) + A^T(i)X - XL(i)C(i) - C^T(i)L^T(i)X + C^T(i)L^T(i)XL(i)C(i) + D^T(i)XD(i).$$

Let $\mathcal{L}(i) \triangleq XL(i)$ and by the Schur complement, (47) implies $\Omega(i) < 0$. Thus,

$$\mathcal{L}\tilde{V}(\hat{x}, e, i) < 0.$$

By [30] we know that the overall closed-loop switched stochastic system composed of the estimation error dynamics (38) and the sliding mode dynamics in the state estimation space (46) is stable. This completes the proof. □

The results of this section have given a full answer to Q3 in Section 1.

5. Illustrative examples

Example 1 (*SMC Problem*)**.** Consider system (1) with $N = 2$ and the following parameters:

Subsystem 1.

$$A(1) = \begin{bmatrix}-0.3 & 0.2 & 0.2\\ 0.3 & -0.1 & 0.3\\ 0.1 & -0.2 & 0.4\end{bmatrix},$$

$$D(1) = \begin{bmatrix}0.7 & 0 & 0.1\\ 0.1 & 0.3 & 0.2\\ 0.3 & 0.1 & 0.4\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 0\\ 2.0\end{bmatrix}, \quad F(1) = 1.6. \tag{55}$$

Subsystem 2.

$$A(2) = \begin{bmatrix}-0.5 & 0.2 & -0.1\\ -0.2 & 0.1 & 0.4\\ 0.3 & 0.1 & 0.3\end{bmatrix},$$

$$D(2) = \begin{bmatrix}0.5 & 0.1 & 0\\ 0.1 & 0.6 & 0.3\\ 0.2 & 0.1 & 0.3\end{bmatrix}, \quad F(2) = 2 \tag{56}$$

and $\alpha = 0.5, f(x, t) = 0.5\exp(-t)\sin(\sqrt{x_1^2 + x_2^2 + x_3^2})$. We checked that the above system with $u(t) = 0$ is unstable for a switching signal given in Fig. 1 (which is generated randomly, here, '1' and '2' represent the first and the second subsystem, respectively). Therefore, our aim is to design a SMC $u(t)$ such that the closed-loop system is mean-square exponentially stable for $T_a > T_a^* = 0.1$ (in this case, the allowable minimum of μ is $\mu_{\min} = 1.0513$). To check the stability of (14) with $T_a > T_a^* = 0.1$ (that is, set $\mu = 1.0513$), we solve (15)–(17), (18) and (19) in Theorem 2 by *Algorithm SMDA*, and obtain

$$\begin{aligned}s(t) &= \mathcal{K}\mathcal{F}^{-1}z_1(t) + z_2(t)\\ &= \begin{bmatrix}1.1775 & 1.4840 & 1.0000\end{bmatrix}x(t).\end{aligned} \tag{57}$$

Now, we will design the SMC of (26)–(27) in Theorem 2. By computation, we have

$$\mathcal{H} \triangleq \begin{bmatrix}\mathcal{K}\mathcal{F}^{-1} & I\end{bmatrix} = \begin{bmatrix}1.1775 & 1.4840 & 1.0000\end{bmatrix}$$
$$\mathcal{H}^+ = \begin{bmatrix}0.2566 & 0.3234 & 0.2179\end{bmatrix}^T, \quad \gamma = 0.4797,$$
$$\phi(1) = 0.8, \quad \phi(2) = 1.0.$$

Thus, the SMC in (26) can be computed as

$$u(t) = \begin{cases} u(t, 1) = -\dfrac{0.4797}{2}s(t) - (0.8 + \delta(1))\text{sign}(s(t)), & i = 1\\ u(t, 2) = -\dfrac{0.4797}{2}s(t) - (1.0 + \delta(2))\text{sign}(s(t)), & i = 2.\end{cases} \tag{58}$$

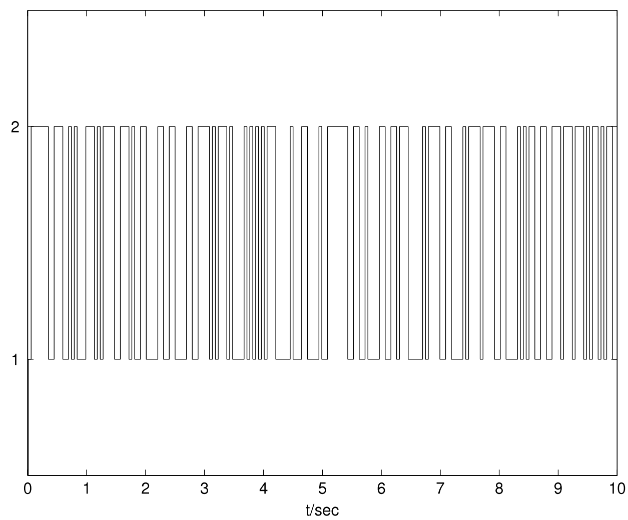

Fig. 1. Switching signal with average dwell time $T_a > T_a^* = 0.1$.

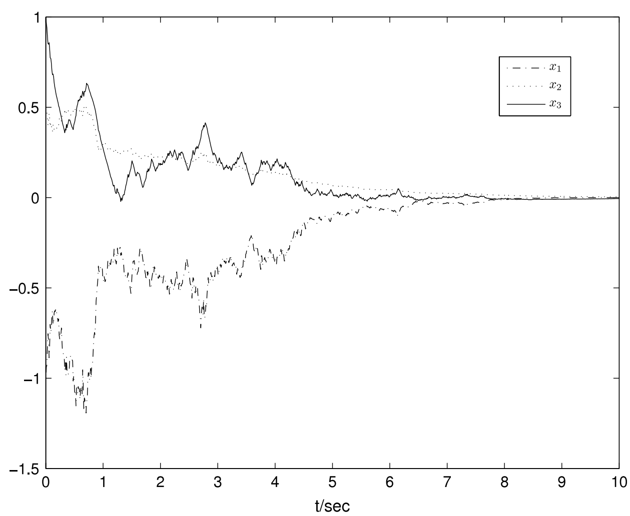

Fig. 2. State of the closed-loop system with (58).

To prevent the control signals from chattering, we replace $\text{sign}(s(t))$ with $s(t)/(0.01 + \|s(t)\|)$. Set $\delta(1) = \delta(2) = 1$ and the initial condition $x(0) = \begin{bmatrix} -1 & 0.5 & 1 \end{bmatrix}^T$. By using the discretization approach in [31], we simulate the standard Brownian motion. Some initial parameters are given as follows: the simulation time $t \in [0, T^*]$ with $T^* = 10$, the normally distributed variance $\delta t = \frac{T^*}{N^*}$ with $N^* = 2^{11}$, step size $\Delta t = \rho \delta t$ with $\rho = 2$, the number of discretized Brownian paths $p = 10$. The simulation results are given in Figs. 2–6. Among them, Figs. 2–4 are the simulation results along an individual discretized Brownian path. Fig. 2 shows the state response of the closed-loop system with (58). The sliding function and the SMC input are given in Figs. 3 and 4, respectively. Figs. 5 and 6 are, respectively, the simulation results on $x(t)$ and $s(t)$ along 10 individual paths (dotted lines) and the average over 10 paths (solid line).

Example 2 (*Observer-Based SMC Problem*)**.** Consider system (1) with $N = 2$ and the following parameters:

Subsystem 1.

$$A(1) = \begin{bmatrix} -0.7 & 0.2 & 0 \\ 0.3 & -0.4 & 0 \\ 0 & 0.4 & 0.2 \end{bmatrix},$$

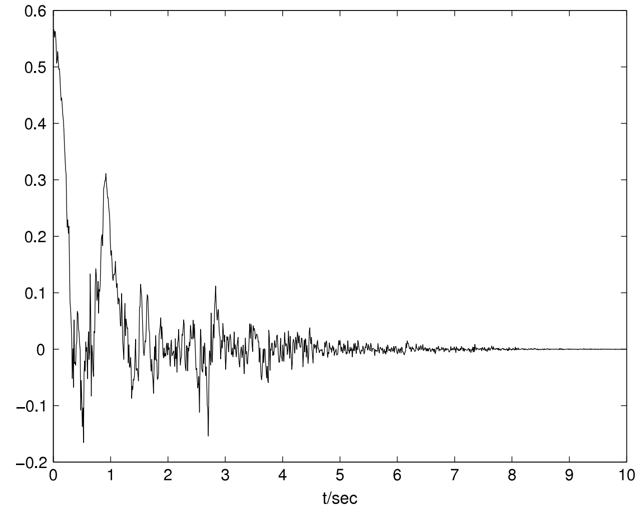

Fig. 3. Sliding function $s(t)$.

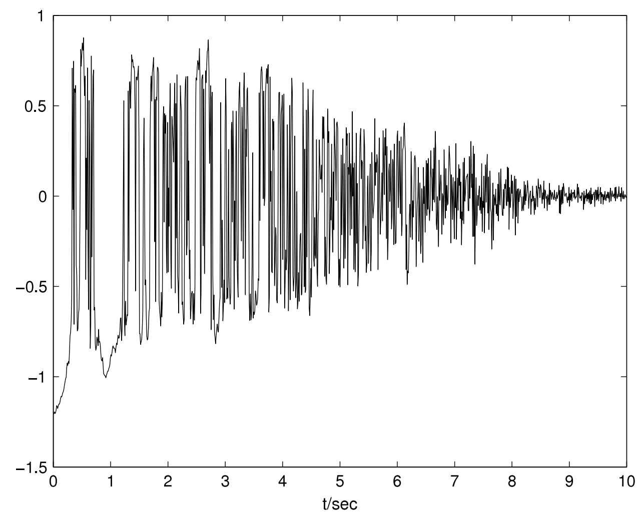

Fig. 4. Control input $u(t)$.

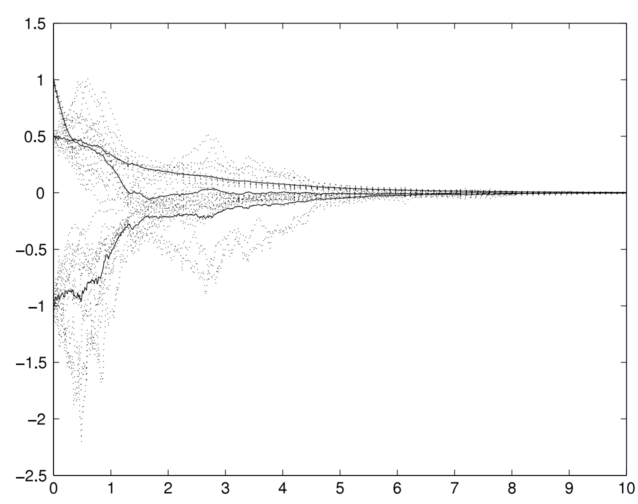

Fig. 5. Individual paths and the average of the state of the closed-loop system with (58).

$$D(1) = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.03 & 0.1 & 0.2 \\ 0 & 0.1 & 0.05 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix},$$

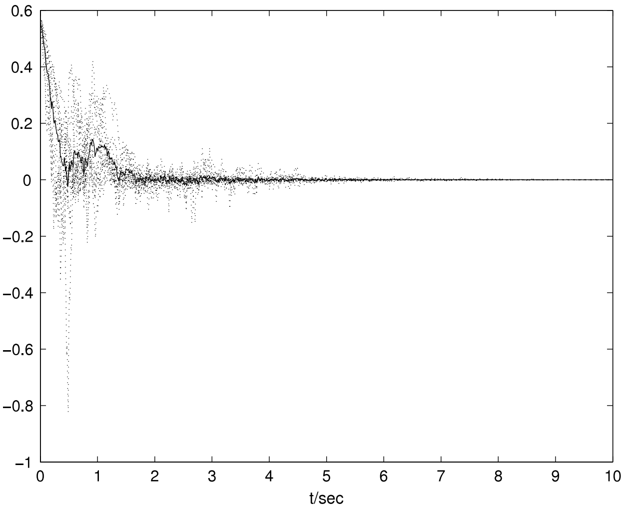

Fig. 6. Individual paths and the average of the sliding function $s(t)$.

$$C(1) = \begin{bmatrix}0.5 & 0.3 & 0.5\end{bmatrix}, \qquad F(1) = \begin{bmatrix}1.0 & 1.0\end{bmatrix}^T. \tag{59}$$

Subsystem 2.

$$A(2) = \begin{bmatrix}-0.5 & 0.2 & 0\\ 0.3 & -0.4 & 0\\ 0 & 0.2 & 0.2\end{bmatrix},$$

$$D(2) = \begin{bmatrix}0.02 & 0.2 & 0\\ 0.03 & 0.1 & 0.1\\ 0 & 0.1 & 0.15\end{bmatrix},$$

$$C(2) = \begin{bmatrix}0.3 & 0.4 & 0.7\end{bmatrix}, \qquad F(2) = \begin{bmatrix}2.0 & 2.0\end{bmatrix}^T,$$

$$f(x,t) = \exp(-t)\sin(\sqrt{x_1^2 + x_2^2 + x_3^2}). \tag{60}$$

In this example, we will consider the SMC design in the case where some of the system state components are not available. According to Section 5, we first design a sliding mode observer in the form of (36) to estimate the system states, and then synthesize an observer-based SMC as in (41)–(42). We select matrix G as follows:

$$G = \begin{bmatrix}0.5 & 2.5 & -2.0\\ 0.3 & -1.5 & 4.0\end{bmatrix}$$

and solving the LMI conditions (47) and (48) in Theorem 4, we have

$$L(1) = \begin{bmatrix}0.3407\\ 0.3131\\ 1.1459\end{bmatrix}, \qquad L(2) = \begin{bmatrix}0.2683\\ 0.1924\\ 0.9101\end{bmatrix}.$$

According to (39)–(40), we have

$$s_x(t) = \begin{bmatrix}0.7363 & 1.7711 & 0.3251\\ 0.6530 & 0.6437 & 1.2172\end{bmatrix}\hat{x}(t) + \sigma(t),$$

$$\dot{\sigma}(t) = \begin{bmatrix}3.2491 & 6.7726 & 3.4232\\ 2.6981 & 2.2973 & 8.6092\end{bmatrix}\hat{x}(t),$$

$$s_e(t) = \begin{bmatrix}0.7363 & 1.7711 & 0.3251\\ 0.6530 & 0.6437 & 1.2172\end{bmatrix}e(t).$$

The state estimate-based SMCs are designed in (41)–(42) with $\phi(1)$ and $\phi(2)$ being $\phi(1) = 1.415$, $\phi(2) = 2.830$, respectively, and

$$\begin{aligned}\chi(t) &= 7.3549 \max_{i\in\mathcal{N}}\{[\|B^T X(A(i) + BG)\| \\ &\quad + \|B^T XL(i)C(i)\|]\|\hat{x}(t)\| + \|B^T XL(i)\|\,\|y(t)\|\}\\ &= 7.3549\max\{(12.5680\|\hat{x}(t)\| + 1.2553\|y(t)\|),\\ &\quad (12.3576\|\hat{x}(t)\| + 0.9349\|y(t)\|)\}.\end{aligned}$$

6. Conclusion

The problem of SMC of a continuous-time switched stochastic system has been investigated in this paper. A sufficient condition for the existence of reduced-order sliding mode dynamics has been derived, and an explicit parametrization of the desired sliding surface has also been given. Then, the SMC for reaching motion has been synthesized. Moreover, we have further studied the observer design and observer-based SMC problems. Some related sufficient conditions have also been established and the observer-based SMC has been synthesized. Numerical examples have been provided to illustrate the effectiveness of the proposed theory.

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