On sparseness, reducibilities, and complexity

Felipe Cucker*

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong

Abstract

We prove several results about the existence of sparse sets complete (and hard) for Turing reductions on different settings and complexity classes over the real numbers.
© 2004 Elsevier B.V. All rights reserved.

1. Introduction

In recent years a number of papers have been published dealing with extensions of Mahaney's Theorem to computations over the real numbers.

Mahaney's Theorem, published in [16], states that, unless P = NP, there are no sparse NP-hard sets. A set $S \subseteq \{0, 1\}^*$ is said to be *sparse* when there is a polynomial p such that for all $n \in \mathbb{N}$ the subset S_n of all elements in S having size n has cardinality at most $p(n)$. Here $\{0, 1\}^*$ denotes the set of all finite sequences of elements in $\{0, 1\}$.

Mahaney's Theorem answers a question which originated from the Berman–Hartmanis conjecture [2]. This conjecture states that all NP-complete sets (over $\{0, 1\}$) are polynomially isomorphic. That is, for all NP-complete sets A and B, there exists a bijection $\varphi : \{0, 1\}^* \to \{0, 1\}^*$ such that $x \in A$ if and only if $\varphi(x) \in B$. In addition both φ and its inverse are computable in polynomial time. So, if the Berman–Hartmanis conjecture holds

---

* Tel.: +852 2784 4208; fax: +852 2788 7446.
  E-mail address: macucker@math.cityu.edu.hk.

then P $\neq$ NP and sparse NP-complete sets do not exist. Mahaney's Theorem shows that these two consequences of the conjecture are equivalent.

After this seminal result of Mahaney, a whole stream of research developed around the issue of reductions to "small" sets (see the surveys [1,5]). Three possible directions in extending Mahaney's Theorem seem to exist: proving Mahaney-type theorems for classes other than NP, for various reducibility types, and for other computation models, in particular, for machines over the reals. This paper is concerned with Mahaney-type theorems for NP, P, and EXP regarding both Turing and many-one reductions for machines over the real numbers.

1.1. On the existence of sparse NP-hard sets

The question of extending Mahaney's Theorem to machines over the real numbers (as introduced in [4]; see also [3]) was first raised in [9]. A first issue that needed to be resolved was how to extend the notion of sparseness to subsets of $\mathbb{R}^\infty$ (the disjoint union of $\mathbb{R}^n$ for $n \in \mathbb{N}$). The notion suggested in [9] is the following. Let $S \subseteq \mathbb{R}^\infty$. We say that S is *sparse* if, for all $n \geq 1$, the set

$$S_n = \{x \in S \mid x \in \mathbb{R}^n\}$$

has dimension at most $\log^q n$ for some fixed q. Here dimension is the dimension, in the sense of algebraic geometry, of the Zariski closure of S_n.

Using this notion of sparseness [9] proves that there are no sparse NP-hard sets under polynomial time many-one reductions in the context of machines over $(\mathbb{R}, +, =)$, i.e., machines which do not perform multiplications or divisions and branch only on equality tests. Note that this result is not conditioned to the inequality P $\neq$ NP since this inequality is known to be true in this setting (cf. [17]).

A natural extension to the result in [9] would consider machines over $(\mathbb{R}, +, \leq)$; that is, machines which do not perform multiplications or divisions but are allowed to branch on inequality tests. While it is unknown whether the existence of sparse NP-hard sets under many-one reductions implies P $=$ NP in this model, Fournier and Koiran [13] show that there exist NP-complete sparse sets with respect to Turing reductions. This follows from a surprising result (Lemma 3 in [13]) which, roughly speaking, states that any NP-complete set over {0, 1} is NP-complete over $(\mathbb{R}, +, \leq)$ for Turing reductions. Since the subsets of elements of size n of any such set S have dimension 0 the sparseness of S is immediate.

This naturally raises the question of whether there exist sparse NP-hard sets over $(\mathbb{R}, +, =)$ with respect to Turing reductions. A partial answer was given by Fournier [12] who shows that there are no sparse definable NP-hard sets over $(\mathbb{R}, +, =)$ with respect to Turing reductions. We recall that $S \subseteq \mathbb{R}^\infty$ is *definable* when, for all $n \geq 1$, S_n is a semialgebraic subset of $\mathbb{R}^n$; that is, S_n is the set of solutions of a Boolean combination of polynomial inequalities. Since any set in NP is definable, an immediate consequence is the following.

Proposition 1. *There are no sparse* NP-*Turing-complete sets over* $(\mathbb{R}, +, =)$.

One cannot hope to remove the requirement that the sparse sets are definable. It is easy to show (see Section 2) that there indeed exist sparse Turing-hard sets for NP in this computational model.

Theorem 1. *There are sparse* NP-*Turing-hard sets over* $(\mathbb{R}, +, =)$.

A model of real machines in which multiplications and divisions are permitted was introduced by Koiran in [15]. To make the model closer to the Turing machine model, the Koiran model heavily penalizes iterated multiplication. This model is called the *weak model*. In the weak model, a machine takes inputs from $\mathbb{R}^{\infty}$ but the cost of computation is measured no longer by the number of arithmetic operations performed by the machine. Instead, the cost of each individual operation $x \circ y$ depends on the sequences of operations which lead to the terms x and y from the input data and the machine constants. For this model, too, it is known that P $\neq$ NP [10].

In [8], it was shown that there are no sparse NP_W-hard sets (with respect to many-one reductions). Here NP_W denotes the class NP for the weak model. The second result in this paper extends Fournier's result to the weak model. Our result is stronger than Fournier's result in that we do not assume definability. Our non-existence result holds for any family of sets satisfying a number of conditions, including the family of definable sets.

Definition 1. Let $\mathscr{F}$ be a family of sets such that every $S \in \mathscr{F}$ is included in $\mathbb{R}^n$ for some $n \geq 1$. We say that $\mathscr{F}$ is *well-behaved* when

  (i) $\mathscr{F}$ contains the semialgebraic sets.
 (ii) $\mathscr{F}$ is closed under finite unions, intersections and complements.
(iii) $\mathscr{F}$ is closed under interior and closure (both interior and closure with respect to the Euclidean topology).
(iv) For all $m, n \geq 1$, for all $U \in \mathscr{F}$, for all rational map $\varphi : U \to \mathbb{R}^m$, and for all $S \in \mathscr{F}$, the following two conditions hold: if $S \subseteq \mathbb{R}^m$ then $\varphi^{-1}(S) \in \mathscr{F}$, and if $S \subseteq U$ then $\varphi(S) \in \mathscr{F}$.
 (v) The notion of dimension is well-defined and it coincides with the usual one for semialgebraic sets. In particular, no set in $\mathscr{F}$ can contain a set of dimension greater than its own or be written as a finite union of sets of smaller dimension.

Let $\mathscr{F}$ be a well-behaved family of sets. We say that sets $S \in \mathscr{F}$ or sets $S \subset \mathbb{R}^{\infty}$ such that $S \cap \mathbb{R}^n \in \mathscr{F}$ for all n are *well-behaved*.

Well-behaved families of sets do exist. The obvious example is the family of semialgebraic sets. But the definition above covers much more general families of sets. In particular, o-minimal structures are well-behaved families (for an overview of o-minimal structures and their geometry see [6] or [19]). Thus, in particular, the family of globally subanalytic sets [11] or the family of sets defined by means of Pfaffian functions [20] are well-behaved.

Theorem 2. *There are no sparse well-behaved* NP_W-*Turing-hard sets. In particular, there are no sparse* NP_W-*Turing-complete sets.*

1.2. On the existence of sparse P-*hard and* EXP-*hard sets*

As pointed above, another direction in which Mahaney's result was extended, in the discrete setting, is the consideration of classes other than NP for completeness (or hardness) results. Particular attention was paid to the class P where a version of the Hartmanis conjecture (also raised by Hartmanis [14]) states that there are no P-complete sparse sets.

In [5] it is noted that "this conjecture, unlike its NP analog, remained open for many years". Actually it remained so until 1995 when Ogiwara [18] proved that if a sparse P-complete set exists then $P \subseteq DSPACE[\log^2 n]$. In this situation, over the reals, we can take advantage of the fact that $NC_{\mathbb{R}} \neq P_{\mathbb{R}}$ holds without assumption even in the unrestricted setting to show the following absolute non-existence result (which includes a trivial extension to $EXP_{\mathbb{R}}$). On its statement, P-completeness is for polylogarithmic parallel time reductions and EXP-completeness is for polynomial parallel time reductions. Also, the word *hard* denotes hardness for many-one reductions and the word *Turing-hard* denotes hardness for Turing reductions.

Theorem 3. (1) *There are no sparse* $P_{\mathbb{R}}$*-hard or* $EXP_{\mathbb{R}}$*-hard sets.*
(2) *There are no sparse well-behaved* $P_{\mathbb{R}}$*-Turing-hard or* $EXP_{\mathbb{R}}$*-Turing-hard sets. In particular, there are no sparse* $P_{\mathbb{R}}$*-Turing-complete or* $EXP_{\mathbb{R}}$*-Turing-complete sets.*
(3) *Both* (1) *and* (2) *hold for the settings* $(\mathbb{R}, +, -, =)$*,* $(\mathbb{R}, +, -, \leq)$*, and the weak model.*

2. Proofs of Theorems 1, 2 and 3

Let us denote by $NP_{add}^{=}$ and $NP_{add}^{\leq}$ the classes of problems in NP over $(\mathbb{R}, +, =)$ and $(\mathbb{R}, +, \leq)$ respectively.

Proof of Theorem 1. Let $\mathscr{S} \subset \{0, 1\}^*$ be any (classical) NP-complete set and consider

$$\mathscr{S}^* = \{(1, x) \mid x \in \mathscr{S}\} \cup \{(2, y) \mid y \in \mathbb{R}, y \geq 0\}.$$

Clearly $\mathscr{S}^*$ is sparse as a subset of $\mathbb{R}^{\infty}$. We now show that it is $NP_{add}^{=}$-Turing-hard. To do so, consider any set $A \in NP_{add}^{=}$. Clearly, $A \in NP_{add}^{\leq}$ as well. But then, Fournier and Koiran [13] show that there is an oracle machine M over $(\mathbb{R}, +, \leq)$ solving A with oracle $\mathscr{S}$ in polynomial time.

We modify M as follows. We replace branch nodes testing a value z for positivity by oracle nodes testing whether $(2, z) \in \mathscr{S}^*$. And we replace oracle nodes testing whether a vector $x \in \mathscr{S}$ by oracle nodes testing whether $(1, x) \in \mathscr{S}^*$. Clearly, the new machine is an oracle machine over $(\mathbb{R}, +, =)$ which, with oracle $\mathscr{S}^*$, decides A in polynomial time.   □

We next proceed to the proof of Theorem 2.
Let $C_n = \{x \in \mathbb{R}^n \mid x_1^{2^n} + \cdots + x_n^{2^n} = 1\}$ and $\mathcal{C} \subset \mathbb{R}^{\infty}$ be the union of the sets C_n. We know that $\mathcal{C} \in NP_W$.

Proposition 2. *Let* $\mathscr{F}$ *be a well-behaved family and let* $S \subset \mathbb{R}^{\infty}$ *be such that* $S \cap \mathbb{R}^n \in \mathscr{F}$ *for all n. Assume S is a* NP_W*-Turing-hard set. Then, there exists* $k \in \mathbb{N}$ *such that, for all* $n \geq 1$*, there exist sets* $E, \Omega \in \mathscr{F}$*,* $E \subset \mathbb{R}^n$ *and* $\Omega \subset C_n$*, and a rational map* $h : \mathbb{R}^n \to \mathbb{R}^m$*, where* $m = n^k$*, well-defined on E and* $\overline{E} \cap \Omega$*, such that*

(i) $E \cap \Omega = \emptyset$,
(ii) $\dim(\overline{E} \cap \Omega) = n - 1$,
(iii) *the degrees of numerator and denominator of the components of h are bounded by a polynomial in n, and*
(iv) $h(\overline{E} \cap \Omega) \cap h(E) = \emptyset$.

Proof. Since $\mathcal{C} \in \mathrm{NP}_W$ and S is NP_W-Turing-hard, there is a deterministic oracle machine M which, with oracle S, decides $\mathcal{C}$ in polynomial time for the weak model. Let p be a polynomial time bound for M.

Consider $n \in \mathbb{N}$. The computation of M over inputs of size n induces a computation tree of depth at most $p(n)$ whose branching nodes are either a sign test or an oracle node.

Let ν be a branching node in this tree. If ν is a sign test, then ν tests whether $\varphi_\nu(x) \geq 0$, where φ_ν is a rational function and $x \in \mathbb{R}^n$ is the input. In addition, since p is a bound for the weak running time of M, both the numerator and denominator of a relatively prime representation of φ_ν have degree bounded by $p(n)$. Note that, since $\mathscr{F}$ is closed under complements and inverse images of rational maps, the sets $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \geq 0\}$ and $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) < 0\}$ are in $\mathscr{F}$.

If instead ν is an oracle node then it tests whether

$$\varphi_\nu(x) = (\varphi_1(x), \ldots, \varphi_m(x)) \in S_m$$

where $m \leq p(n)$ and, for $i = 1, \ldots, m$, φ_i is a rational function as above. Again, since $\mathscr{F}$ is closed under complements and inverse images of rational maps, the sets $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \in S_m\}$ and $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \notin S_m\}$ are in $\mathscr{F}$.

For any leaf γ in the tree, we denote by Ω_γ the set of points in $\mathbb{R}^n$ whose computation ends in γ. The set Ω_γ is the intersection of sets of the form $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \geq 0\}$ or $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) < 0\}$ with ν a branching node, and sets of the form $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \in S_m\}$ or $\{x \in \mathbb{R}^n \mid \varphi_\nu(x) \notin S_m\}$ with ν an oracle node. Since in all four cases these sets are in $\mathscr{F}$ we conclude that $\Omega_\gamma \in \mathscr{F}$. Now let $\mathcal{A}$ be the set of accepting leaves. Then,

$$C_n = \bigcup_{\gamma \in \mathcal{A}} \Omega_\gamma.$$

Since $\dim(C_n) = n - 1$ and the union above is a finite union of sets in $\mathscr{F}$ there exists a leaf $\gamma^0 \in \mathcal{A}$ such that $\dim \Omega_{\gamma^0} = n - 1$. So, Ω_{γ^0} is a subset of C_n, it belongs to $\mathscr{F}$, and it is of maximal dimension (among the Ω_γ for $\gamma \in \mathcal{A}$).

Let ν be a branching node in the path leading to γ^0. The *domain* of ν is

$$\Omega_\nu = \{x \in \mathbb{R}^n \mid x \text{ reaches the node } \nu\}$$

and its *excluded part*,

$$E_\nu = \{x \in \Omega_\nu \mid x \text{ deviates at } \nu \text{ from the path leading to } \gamma^0\}.$$

If $\nu_1, \ldots, \nu_\ell$ are the branching nodes in the path leading to γ^0 we then have the disjoint union

$$\mathbb{R}^n - \Omega_{\gamma^0} = E_{\nu_1} \cup \cdots \cup E_{\nu_\ell}.$$

Again, we remark that $E_{\nu_1}, \ldots, E_{\nu_\ell}$ are all sets in $\mathscr{F}$.

We next show that there exists $i \leq \ell$ such that $\dim(\overline{E_{\nu_i}} \cap \Omega_{\gamma^0}) = n - 1$.

This follows from the fact that, since taking closures commutes with finite unions,

$$\mathbb{R}^n = \overline{E_{\nu_1}} \cup \cdots \cup \overline{E_{\nu_\ell}}.$$

Therefore,

$$\Omega_{\gamma^0} = (\Omega_{\gamma^0} \cap \overline{E_{\nu_1}}) \cup \cdots \cup (\Omega_{\gamma^0} \cap \overline{E_{\nu_\ell}})$$

and, since $\dim \Omega_{\gamma^0} = n - 1$, it follows that there exists $i \leq \ell$ such that $\dim(\Omega_{\gamma^0} \cap \overline{E_{\nu_i}}) = n - 1$.

Let $E = E_{\nu_i}$, $\Omega = \Omega_{\gamma^0}$, and $h = \varphi_{\nu_i}$, $h : \mathbb{R}^n \to \mathbb{R}^{m_i}$. We just proved that $\dim(\Omega \cap \overline{E}) = n - 1$ and thus (ii) holds. In addition, $\Omega_{\gamma^0} \subseteq \Omega_{\nu_i}$, from which $E \cap \Omega = \emptyset$ and (i) holds as well. Part (iii) follows, as we already remarked, from the weakness of M. Finally, for part (iv), consider first the case that ν_i is an oracle node. Then for $\mathcal{S}$ either S_{m_i} or its complement, we have $E_{\nu_i} \subseteq \varphi_{\nu_i}^{-1}(\mathcal{S})$ and $\Omega_{\gamma^0} \subseteq \varphi_{\nu_i}^{-1}(\mathcal{S}^c)$, where [c] denotes complement, and therefore $h(\overline{E} \cap \Omega) \cap h(E) = \emptyset$. A similar reasoning holds if ν_i is a test node with $\mathcal{S}$ now either $\mathbb{R}^+$ or $\mathbb{R}^- - \{0\}$, showing that in this case as well, $h(\overline{E} \cap \Omega) \cap h(E) = \emptyset$. $\square$

The following result in real algebraic geometry will be used. Its proof can be found in Chapter 19 of [3].

Proposition 3. *Let* $f \in \mathbb{R}[x_1, \ldots, x_n]$ *be an irreducible polynomial such that the dimension of its zero set* $\mathcal{Z}(f) \subseteq \mathbb{R}^n$ *is* $n - 1$. *Then, for any polynomial* $g \in \mathbb{R}[x_1, \ldots, x_n]$, g *vanishes on* $\mathcal{Z}(f)$ *if and only if* g *is a multiple of* f. $\square$

Let $f_n = x_1^{2^n} + \cdots + x_n^{2^n} - 1$ so that $C_n = \{x \in \mathbb{R}^n \mid f_n(x) = 0\}$.

Proposition 4. *With the notations of Proposition 2, let* $k - 1 = \dim h(\overline{E} \cap \Omega)$.

(i) *There exist indices* $i_1, \ldots, i_k \in \{1, \ldots, m\}$, *a polynomial* $g \in \mathbb{R}[y_1, \ldots, y_k]$ *and a rational function* $q \in \mathbb{R}(x_1, \ldots, x_n)$ *with both numerator and denominator relatively prime to* f_n *such that*

$$g(h_{i_1}, \ldots, h_{i_k}) = f_n^\ell q$$

for some $\ell > 0$.
(ii) *For* n *sufficiently large,* $k \geq n$.

Proof. Let $K = \dim h(\overline{E})$. Since $\dim h(E) = K$, there exist $i_1, \ldots, i_K \in \{1, \ldots, m\}$ such that the functions $h_{i_1}, \ldots, h_{i_K}$ are algebraically independent.

We next want to show that $k \leq K$. To do so let $X = h(\overline{E} \cap \text{dom}(h))$, $Y = h(E)$ and $Z = h(\overline{E} \cap \Omega)$. Here $\text{dom}(h)$ denotes the set of points in $\mathbb{R}^n$ where h is well-defined. We have that all X, Y and Z are sets of $\mathscr{F}$ in $\mathbb{R}^m$. In addition, Z is contained in the closure of Y with respect to the Euclidean topology relative to X since h is continuous and $Y \cap Z = \emptyset$ by Proposition 2(iv). From here it follows that Z is included in the boundary of Y relative to X. Hence, $\dim Z < \dim Y = \dim X$.

The above shows that $\dim h(\overline{E} \cap \Omega) < K$, i.e. $k \leq K$. Therefore, there exists a set of k elements in $\{i_1, \ldots, i_K\}$, which we may assume are $i_1, \ldots, i_k$, and a polynomial $g \in \mathbb{R}[y_1, \ldots, y_k]$ such that, for all $x \in \overline{E} \cap \Omega$, $g(h_{i_1}(x), \ldots, h_{i_k}(x)) = 0$. Write this as a rational function $g(h) = a/b$ with $a, b \in \mathbb{R}[x_1, \ldots, x_n]$ relatively prime. Then, since $\dim(\overline{E} \cap \Omega) = n - 1$, $\overline{E} \cap \Omega \subset C_n$, C_n is irreducible and $h_{i_1}, \ldots, h_{i_k}$ are algebraically

independent, $a(C_n) = 0$ and $a \neq 0$. By Proposition 3 this implies that there exists $r \in \mathbb{R}[x_1, \ldots, x_n]$ such that $a = rf_n$. If ℓ is the largest power of f_n dividing a then part (i) follows by taking $q = \frac{r'}{b}$, where r' is the quotient of r divided by $f_n^{\ell-1}$.

Part (ii) is proved as in Proposition 3.3 of [8]. $\quad \square$

Proof of Theorem 2. Assume the set S in Proposition 2 is sparse and let q be a polynomial such that $\dim(S_n) \leq q(\log n)$. Let $n \in \mathbb{N}$, $n \geq 3$, be sufficiently large such that $q(\log p(n)) < n - 1$ and part (ii) of Proposition 4 holds. Recall that p is a polynomial bounding the running time of the reduction in Proposition 2.

Recall from the proof of Proposition 2 that $h = \varphi_\nu$ for some branching node ν in the tree associated to M and n. First assume that ν is a sign test. Then $\dim(h(\Omega)) \leq 1$ since $h(\Omega) \subseteq \mathbb{R}$. But

$$\dim h(\Omega) \geq \dim h(\overline{E} \cap \Omega) = k - 1 \geq n - 1 > 1.$$

Therefore, ν can not be a sign test and is an oracle node instead. Let m be such that $h : \mathbb{R}^n \to \mathbb{R}^m$. Then $m \leq p(n)$ and

$$\dim h(\Omega) \geq \dim h(\overline{E} \cap \Omega) = k - 1 \geq n - 1 > q(\log p(n)) \geq \dim S_m$$

and

$$\dim h(E) = K \geq k \geq n > q(\log p(n)) \geq \dim S_m.$$

This is a contradiction since either $h(E)$ or $h(\Omega)$ is included in S_m. $\quad \square$

We finally proceed with the proof of Theorem 3. Before doing so, we recall that, for $k \in \mathbb{N}$, $\mathrm{NC}_\mathbb{R}^k$ denotes the class of subsets of $\mathbb{R}^\infty$ which can be decided in parallel time $\mathcal{O}(\log^k n)$. The class $\mathrm{NC}_\mathbb{R}$ is defined to be the union of the classes $\mathrm{NC}_\mathbb{R}^k$. Also, $\mathrm{PAR}_\mathbb{R}$ is defined to be the class of subsets of $\mathbb{R}^\infty$ which can be decided in parallel time $n^{\mathcal{O}(1)}$. Functional versions $\mathrm{FNC}_\mathbb{R}$ and $\mathrm{FPAR}_\mathbb{R}$ of $\mathrm{NC}_\mathbb{R}$ and $\mathrm{PAR}_\mathbb{R}$ are defined in the obvious way. For formal definitions and basic properties of these classes see Chapter 18 in [3]. When we refer to $\mathrm{P}_\mathbb{R}$-hardness, reductions are in $\mathrm{FNC}_\mathbb{R}$. Similarly, when talking about $\mathrm{EXP}_\mathbb{R}$-hardness, reductions are in $\mathrm{FPAR}_\mathbb{R}$.

Proof of Theorem 3. In [7] it is shown that the set $\mathcal{C}$ defined just before Proposition 2 does not belong to $\mathrm{NC}_\mathbb{R}$. The idea behind the proof is to assume it does and to consider the computation tree obtained by unwinding the $\mathrm{NC}_\mathbb{R}$ computation. This tree has an enormous number of leaves since its branching nodes may have a polynomial number of successors. But the functions computed along its paths (and in particular those whose sign is tested at the branching nodes) are rational functions whose numerators and denominators have degrees polynomially bounded.

Let now φ be an $\mathrm{FNC}_\mathbb{R}$ reduction from $\mathcal{C}$ to a $\mathrm{P}_\mathbb{R}$-hard set S. Using the above, the proof of Proposition 3.2 in [8] can be adapted to hold for φ. This yields, for all n sufficiently large, a point $x \in C_n$ and an open ball $U \subseteq \mathbb{R}^n$ centered at x such that the restriction of φ to U is a rational map $h : U \to \mathbb{R}^m$ for some m bounded by a polynomial in n. In addition, if $h_1, \ldots, h_m$ are the coordinates of h, then the degrees of numerator and denominator of h_i are bounded by a polynomial in n, for $i = 1, \ldots, m$.

From the above, the arguments in [8, Proposition 3.3] yield a version of Proposition 4 with $\overline{E} \cap \Omega$ replaced by $U \cap C_n$. The rest of the proof of (1) for $P_{\mathbb{R}}$ easily follows.

The proof for $\text{EXP}_{\mathbb{R}}$ is similar except that, for $n \geq 1$, we take

$$C_n = \{x \in \mathbb{R}^n \mid x_1^{2^{2^n}} + \cdots + x_n^{2^{2^n}} = 1\}.$$

The proof of (2) is done in the same manner. One now uses a simple modification in the proof of Proposition 2.

Also the proof of (3) is similar. Note that in the additive setting one has to replace degree by coefficient size. Thus, for instance, over either $(\mathbb{R}, +, -, \leq)$ or $(\mathbb{R}, +, -, =)$, the set $\mathcal{C}$ given by

$$C_n = \{x \in \mathbb{R}^n \mid 2^n x_1 + \cdots + 2^n x_n = 1\}$$

is in P but not in NC. $\square$

Acknowledgement

References

[1] V. Arvind, Y. Han, L. Hemachandra, J. Köbler, A. Lozano, M. Mundhenk, M. Ogiwara, U. Schöning, R. Silvestri, T. Thierauf, Reductions to sets of low information content, in: K. Ambos-Spies, S. Homer, U. Schöning (Eds.), Complexity Theory: Current Research, Cambridge University Press, 1993, pp. 1–45.

[2] L. Berman, J. Hartmanis, On isomorphism and density of NP and other complete sets, SIAM J. Comput. 6 (1977) 305–322.

[3] L. Blum, F. Cucker, M. Shub, S. Smale, Complexity and Real Computation, Springer-Verlag, 1998.

[4] L. Blum, M. Shub, S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, Bull. Amer. Math. Soc. 21 (1989) 1–46.

[5] J.-Y. Cai, M. Ogihara, Sparse sets versus complexity classes, in: L.A. Hemaspandra, A.L. Selman (Eds.), Complexity Theory Retrospective II, Springer-Verlag, 1997, pp. 53–80.

[6] M. Coste, An Introduction to o-minimal Geometry, Istituti Editoriali e Poligrafici Internazionali, 2000.

[7] F. Cucker, $P_{\mathbb{R}} \neq NC_{\mathbb{R}}$, J. Complexity 8 (1992) 230–238.

[8] F. Cucker, D.Yu. Grigoriev, There are no sparse NP_W-hard sets, SIAM J. Comput. 31 (2001) 193–198.

[9] F. Cucker, P. Koiran, M. Matamala, Complexity and dimension, Inform. Process. Lett. 62 (1997) 209–212.

[10] F. Cucker, M. Shub, S. Smale, Separation of complexity classes in Koiran's weak model, Theoret. Comput. Sci. 133 (1994) 3–14.

[11] J. Denef, L. van den Dries, p-adic and real subanalytic sets, Ann. of Math. 128 (1998) 80–138.

[12] H. Fournier, Sparse NP-complete problems over the reals with addition, Theoret. Comput. Sci. 255 (2001) 607–610.

[13] H. Fournier, P. Koiran, Lower bounds are not easier over the reals: inside PH, in: 28th International Colloquium on Automata, Languages and Programming, Lect. Notes in Comp. Sci., vol. 1853, Springer-Verlag, 2000, pp. 832–843.

[14] J. Hartmanis, On log-tape isomorphisms of complete sets, Theoret. Comp. Sci. 7 (1978) 231–243.

[15] P. Koiran, A weak version of the Blum, Shub & Smale model, J. Comput. System Sci. 54 (1997) 177–189, A preliminary version appeared in 34th Annual IEEE Symp. on Foundations of Computer Science, pp. 486–495, 1993.

[16] S.R. Mahaney, Sparse complete sets for NP: solution of a conjecture by Berman and Hartmanis, J. Comput. System Sci. 25 (1982) 130–143.

[17] K. Meer, A note on a P $\neq$ NP result for a restricted class of real machines, J. Complexity 8 (1992) 451–453.
[18] M. Ogihara, Sparse sets for P yield space-efficient algorithms, in: 36th Annual IEEE Symp. on Foundations of Computer Science, 1995, pp. 354–361.
[19] L. van den Dries, C. Miller, Geometric categories and o-minimal structures, Duke Math. J. 84 (1996) 497–540.
[20] A.J. Wilkie, A theorem of the complement and some new o-minimal structures, Selecta Math. (N.S.) 5 (1999) 397–421.