Tridiagonal Matrices and Trees

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Tridiagonal matrices and three term recurrences and second order equations appear amazingly often, throughout all of mathematics. We won’t try to review this subject. Instead we look in two less familiar directions.

Here is a tridiagonal matrix problem that waited surprisingly long for a solution. Forward elimination factors $T$ into $LDU$, with the pivots in $D$ as usual. Backward elimination, from row $n$ to row 1, factors $T$ into $U_D L_D$. Parlett asked for a proof that $\text{diag}(D + D') = \text{diag}(T) + \text{diag}(T^{-1})^{-1}$. In an excellent paper (Lin Alg Appl 1997) Dhillon and Parlett extended this four-diagonal identity to block tridiagonal matrices, and also applied it to their “Holy Grail” algorithm for the eigenproblem. I would like to make a different connection, to the Kalman filter.

The second topic is a generalization of tridiagonal to “tree-diagonal”. Unlike the interval, the tree can branch. In the matrix $T$, each vertex is connected only to its neighbors (but a branch point has more than two neighbors). The continuous analogue is a second order differential equation on a tree. The “non-jump” conditions at a meeting of $N$ edges are continuity of the potential ($N-1$ equations) and Kirchhoff’s Current Law (1 equation). Several important properties of tridiagonal matrices, including $O(N)$ algorithms, survive on trees.