Spectrum of Convolution Dilation Operators on Weighted $L^p$ Spaces

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The convolution dilation operators

$$W_{c,\alpha} f(x) = \alpha \int_{\mathbb{R}} c(\alpha x - y) f(y) dy, \quad f \in L^p(\mathbb{R}),$$

where $\alpha > 1$, for special integrable kernels $c$ arise in many diverse disciplines, such as wavelet analysis, geometric modelling, dynamical systems and the mathematical theory of quasicrystals. Their spectrum and eigenfunctions play an important role in applications. The object here is to give a complete understanding of the spectrum of the operator for any compactly supported integrable kernel $c$ in the setting of the weighted $L^p$ space, $L^p_w(\mathbb{R})$ that comprises functions $f$ for which $f w \in L^p(\mathbb{R})$. We prove that under an oscillation condition on $w$, $W_{c,\alpha}$ is a compact operator on $L^p_w(\mathbb{R})$ if and only if

$$\lim_{|x| \to \infty} w(x)/w(\alpha x) = 0.$$  \(\text{We also prove that if }\lim_{|x| \to \infty} w(x)/w(\alpha x) = r\text{ for some positive constant }r\text{, then the spectrum of }W_{c,\alpha}\text{ on the space }L^p_w(\mathbb{R})\text{ is the closed disc }D_r := \{\lambda \in \mathbb{C} : |\lambda| \leq r \alpha^{1-1/p}\}\text{ in addition to the set }\{\alpha^{-k} : k = 1, 2, \ldots\}\text{, and that all nonzero complex numbers with absolute value strictly less than }r\text{ are eigenvalues of the operator }W_{c,\alpha}\text{ on }L^p_w(\mathbb{R}).\text{ In particular, for }w = 1\text{ the results say that the spectrum of }W_{c,\alpha}\text{ on }L^p(\mathbb{R})\text{ is the closed disc with centre at the origin and radius }\alpha^{1-1/p}\text{, and that all nonzero complex numbers with absolute value strictly less than 1 are its eigenvalues.}