Some Results on Wavelet Frames

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There have been many results for the system \(\{a^{-m/2}\psi(a^{-m}x - nb)\}_{m,n \in \mathbb{Z}}\) to be a frame for \(L^2(R)\), where \(a > 1\) and \(b > 0\). In this paper we study some general conditions for the system \(\psi_j(x) = s_j^{-1/2}\psi(s_j^{-1}x - b_k), j, k \in \mathbb{Z}\), to be a frame, where \(s_j > 0\) and \(b_k\) are reals. First we give a necessary condition which extensively extends the one obtained by Chui and Shi in 1993. Secondly we point out that the necessary condition is also sufficient when \(\psi\) is bandlimited, moreover we give sufficient conditions, which exclusively depend on \(\psi\), for \(\{\psi_j\}\) to be a frame for a large class of \(\{s_j\}\) and \(\{b_k\}\).

The main results of this paper are as follows.

**Theorem 1.** If \(s_j^{-1/2}\psi(s_j^{-1}x - b_k), j, k \in \mathbb{Z}\), is a frame with bounds \(A > 0\) and \(B < \infty\), where \(\psi \in L^2(R)\), \(s_j > 0\) and \(\{b_k\}\) is a r-Fourier frame sequence, then

\[
\frac{A}{B} \leq \sum_{j \in \mathbb{Z}} |\psi(s_j w)|^2 \leq \frac{B}{A} \quad a.e.
\]

**Theorem 2.** Suppose \(\psi \in B_{\Omega} \cap L(R)\) for some \(\Omega > 0\), where \(B_{\Omega} = \{f \in L^2(R) : \hat{f}(w) = 0\ \text{for any } |w| > \Omega\}\), and \(s_j > 0\). Then the following two assertions are equivalent: (i) There is a r-Fourier frame sequence \(\{b_k\}\) with \(r > 2\Omega\) such that \(\psi_j(x) = s_j^{-1/2}\psi(s_j^{-1}x - b_k), j, k \in \mathbb{Z}\), is a frame for \(L^2(R)\); (ii) \(a \leq \sum_{j \in \mathbb{Z}} |\psi(s_j w)|^2 \leq b\), a.e., for some \(a > 0\) and \(b < \infty\). Furthermore if (ii) is satisfied, then \(\{\psi_j\}\) is a frame for any r-Fourier frame sequence \(\{b_k\}\), where \(r > 2\Omega\).

**Theorem 3.** Suppose \(\psi \in B_{\Omega} \cap L(R)\) for some \(\Omega > 0\). If \(|\hat{\psi}(w)| \leq A|w|^\alpha\) near \(w = 0\) for some \(A > 0\) and \(\alpha > 0\) and 0 is an isolated zero of \(\hat{\psi}(w)\), then \(\{s_j^{-1/2}\psi(s_j^{-1}x - b_k)\}_{j,k \in \mathbb{Z}}\) is a frame for \(L^2(R)\) for any sequence \(\{s_j\}_{j \in \mathbb{Z}}\) of positives satisfying \(0 < \inf \frac{s_j}{s_{j+1}} < \sup \frac{s_j}{s_{j+1}} < 1\) and any r-Fourier frame sequence \(\{b_k\}_{k \in \mathbb{Z}}\), where \(r > 2\Omega\).