Rotation Invariant Multiresolutions and Cardinal Polysplines

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A cardinal polyspline of the order $p$ and of scaling parameter $j \in \mathbb{Z}$ is by definition a $(2p-2)$-times continuously differentiable function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, which is polyharmonic of the order $p$ on every open annulus $A_{l,j} := \{ x \in \mathbb{R}^n : \epsilon^{l2^{-j}} < |x| < \epsilon^{(l+1)2^{-j}} \}$ for $l \in \mathbb{Z}$, i.e. it is a solution of the equation $\Delta^p = 0$ on each such annulus where $\Delta$ is the Laplace operator.

In analogy to the univariate case, scaling spaces $V_j$ are defined as the $L^2$-closure of the set of all square-integrable cardinal polysplines of order $p$ and of scaling parameter $j$. The sequence $(V_j)_{j \in \mathbb{Z}}$ is a multiresolution of (rotation invariant) subspaces of $L^2(\mathbb{R}^n)$ in the sense of deBoor, Devore, Ron, i.e., it is nested, the intersection of the spaces $V_j$ is trivial and the union is dense. The main result is the following: the decomposition of $L^2(\mathbb{R}^n)$ through spherical harmonics leads to a decomposition of the wavelet space $W_j$, defined by $W_j := V_{j+1} \oplus V_j$, as an orthogonal sum of univariate wavelet spaces $W_j(k,l)$ where $k \in \mathbb{N}_0$ denotes the degree of the spherical harmonic, $a_k$ is the dimension of the space of all spherical harmonics of degree $k$, and finally $l = 1, ..., a_k$. Moreover, the wavelet space $W_j(k,l)$ is generated by translations of a single wavelet function. In the framework of nonstationary wavelet analysis one can say that the father wavelet of the mother wavelet is the basic $L$-spline of a suitable linear differential operator $L_k$ with constant coefficients.