The State of Art in Spectral Methods
Mathematics Subject Classification.
65M99; Secondary 65N99.
The State of Art in Spectral Methods

by Guo Ben-yu

Lecture Notes in Mathematics
It is the policy of the Centre for Mathematical Sciences of the City University of Hong Kong to publish lecture series given by eminent scholars working in or visiting the Centre. The lecture notes are aimed primarily at Research Students and Mathematicians who are non-experts in a particular area.
Contents

CONTENTS ................................................................. v

Preface ............................................................... vii

1 Introduction ....................................................... 1

2 Orthogonal Approximations in Sobolev Spaces .......... 7
   2.1 Preliminaries ................................................. 7
   2.2 Fourier Approximations .................................. 9
   2.3 Orthogonal Systems of Polynomials In A Finite Interval .... 12
   2.4 Legendre Approximations ................................. 15
   2.5 Chebyshev Approximations ............................... 19
   2.6 Some Orthogonal Approximations In Infinite Intervals .... 23
   2.7 Filterings And Recovering The Spectral Accuracy ....... 27

3 Stability and Convergence ....................................... 34
   3.1 Stability And Convergence For Linear Problems ......... 34
   3.2 Generalized Stability For Nonlinear Problems .......... 36
   3.3 Initial-Value Problems .................................... 40

4 Spectral Methods and Pseudospectral Methods ........... 43
   4.1 Fourier Spectral Methods and Fourier Pseudospectral Methods .... 43
   4.2 Legendre Spectral Methods And Legendre Pseudospectral Methods .... 48
   4.3 Chebyshev Spectral Methods And Chebyshev Pseudospectral Methods .... 54
   4.4 Spectral Penalty Methods .................................. 57
   4.5 Spectral Vanishing Viscosity Methods .................. 61
   4.6 Spectral Approximations of Isolated Solutions .......... 64

5 Spectral Methods for Multi-dimensional and High Order Problems 67
   5.1 Orthogonal Approximations In Several Dimensions ....... 67
   5.2 Spectral Methods For Multi-Dimensional Nonlinear Systems .... 73
   5.3 Spectral Methods For Nonlinear High Order Equations ....... 79
   5.4 Spectral Domain Decomposition Methods .................. 84
   5.5 Spectral Multigrid Methods ................................ 91

6 Mixed Spectral Methods .......................................... 97
   6.1 Mixed Fourier-Legendre Approximations .................. 97
   6.2 Mixed Fourier-Chebyshev Approximations ................. 100
   6.3 Applications ............................................... 103
7 Combined Spectral Methods
    7.1 Some Basic Results In Finite Element Methods .................. 107
    7.2 Combined Fourier-Finite Element Approximations .............. 108
    7.3 Combined Legendre-Finite Element Approximations ........... 110
    7.4 Combined Chebyshev-Finite Element Approximations .......... 112
    7.5 Applications ............................................. 114

8 Spectral Methods on the Spherical Surface  120
    8.1 Spectral Approximation On The Spherical Surface ........... 120
    8.2 Pseudospectral Approximation On The Spherical Surface .... 122
    8.3 Applications ............................................... 123

References  127
Preface

These lecture notes are based on the eight lectures on spectral methods and their applications at the City University of Hong Kong in 1996. Their purpose is to present the basic algorithms, the main theoretical results and some applications of spectral methods.

The outline of these lecture notes is as follows. Lecture 1 is a colloquial introduction to spectral methods. In Lecture 2, we discuss various orthogonal approximations in Sobolev spaces. We also discuss the filterings and recovering the spectral accuracy. Lecture 3 is a survey of the theory of stability and convergence. Lecture 4 consists of two parts. In the first part, we present some basic spectral methods with their applications to nonlinear problems. In the second part, we consider the spectral penalty methods, the spectral viscosity methods and the spectral approximations of isolated solutions. Lecture 5 is devoted to the spectral approximations of multi-dimensional and high order problems. The spectral domain decomposition methods and the spectral multigrid methods are also introduced. We consider the mixed spectral methods for semi-periodic problems in Lecture 6, and some combined spectral methods in Lecture 7. The final lecture focuses on the recent developments of spectral methods on the spherical surface.

Guo Ben-yu
Lecture 1

Introduction

In the past two decades, spectral methods have developed rapidly. They have been applied successfully to numerical simulations in many fields, such as heat conduction, fluid dynamics, quantum mechanics and so on.

The main feature of the spectral methods is to take various orthogonal systems of infinitely differentiable global functions as trial functions. Its fascinating merit is the high accuracy, the so-called convergence of “infinite order”. For example, we consider numerical solution of the Laplace equation on a cube, and divide the domain into $N^3$ uniform subcubes. If we use linear finite element methods, then the error in the $L^2$-norm between the genuine solution and the numerical one is of order $N^{-2}$, no matter how smooth the genuine solution is. If we use the standard “nine points” difference scheme with the uniform mesh size $\frac{1}{N}$, then the error is of the same order. However, the smoother the genuine solution, the higher the convergence rate of the numerical solution by spectral methods. In particular, when the genuine solution is infinitely differentiable, the numerical one converges faster than $N^{-\alpha}$, $\alpha$ being any positive constant.

The basic idea of spectral methods stems from Fourier analysis. In 1820, Navier considered a thin plate problem governed by the equation

$$\frac{\partial^4 U}{\partial x^4} + 2 \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = f, \quad 0 < x, y < \pi, \quad (1.1)$$

with the boundary conditions

$$U = \frac{\partial^2 U}{\partial x^2} = 0, \quad x = 0, \pi,$$

$$U = \frac{\partial^2 U}{\partial y^2} = 0, \quad y = 0, \pi.$$

Suppose that $f(x, y)$ can be expanded as

$$f(x, y) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} f_{l,m} \sin lx \sin my.$$

We look for the solution as

$$U(x, y) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} U_{l,m} \sin lx \sin my. \quad (1.2)$$
By substituting (1.2) into (1.1), we deduce that

$$U_{i,m} = \frac{f_{i,m}}{t^4 + 2t^2m^2 + m^4}.$$  

Finally we truncate (1.2) to obtain the approximate solution

$$u_N(x,y) = \sum_{i=1}^{N} \sum_{m=1}^{N} U_{i,m} \sin lx \sin my$$

which is the solution of Fourier spectral method for problem (1.1).

There are several kinds of spectral methods, called spectral methods, pseudospectral methods and Tau methods, respectively. All of them can be derived from the method of weighted residuals. Let \( \Omega \) be a spatial domain with the boundary \( \partial \Omega \), and \( \hat{\Omega} = \Omega \cup \partial \Omega \). We consider here an initial-boundary value problem as follows

$$\begin{align*}
    LU = f, & \quad x \in \Omega, t > 0, \\
    BU = 0, & \quad x \in \partial \Omega, t > 0, \\
    U(x,0) = U_0(x), & \quad x \in \hat{\Omega}
\end{align*}$$

(1.3)

where \( L \) is a differential operator and \( B \) is a linear boundary operator, \( f(x,t) \) and \( U_0(x) \) are given functions. Assume that some conditions are fulfilled for ensuring the existence, uniqueness and regularity of solution of (1.3). The method of weighted residuals is to find an approximate solution of the form

$$u_N(x,t) = u_B(x,t) + \sum_{i=0}^{N} u_{N,i}(t) \phi_i(x)$$

(1.4)

where the trial functions \( \phi_i(x), 0 \leq i \leq N \) are linearly independent, \( u_B(x,t) \) is chosen in such a way that \( u_N(x,t) \) fits the boundary conditions. The unknown coefficients \( u_{N,i}(t), 0 \leq i \leq N \), are determined by the equation

$$\begin{align*}
    \int_{\Omega} (LU_N(x,t) - f(x,t)) \psi_j(x) \, dx = 0, & \quad 0 \leq j \leq N, t > 0, \\
    \int_{\Omega} (u_N(x,0) - U_0(x)) \psi_j(x) \, dx = 0, & \quad 0 \leq j \leq N
\end{align*}$$

(1.5)

where the weight functions \( \psi_j(x), 0 \leq j \leq N \), are linearly independent.

We first derive the spectral methods. In this case, each \( \phi_i(x) \) satisfies the boundary conditions and so \( u_B(x,t) \equiv 0 \). In addition, \( \psi_j(x) = \phi_j(x) \). For simplicity, we introduce the inner product and the norm of the space \( L^2(\Omega) \), i.e., for any \( v, w \in L^2(\Omega) \),

$$\langle v, w \rangle = \int_{\Omega} v(x) w(x) \, dx, \quad ||v|| = (v, v)^{\frac{1}{2}}.$$  

Accordingly, (1.5) stands for

$$\begin{align*}
    \langle LU_N(t), \phi_l \rangle = \langle f(t), \phi_l \rangle, & \quad 0 \leq l \leq N, t > 0, \\
    \langle u_N(0), \phi_l \rangle = \langle U_0, \phi_l \rangle, & \quad 0 \leq l \leq N.
\end{align*}$$

(1.6)
It is more convenient to describe scheme (1.5) by a projection operator \( P_N \). To do this, we set a finite dimensional space \( V_N \) as
\[
V_N = \text{span}\{\phi_l \mid 0 \leq l \leq N\}
\]
which is called the trial function space. For any \( v \in L^2(\Omega) \), the \( L^2 \)-projection \( P_N v \in V_N \), satisfies
\[
(v - P_N v, \phi_l) = 0, \quad 0 \leq l \leq N.
\]
Since \( \phi_l(x), 0 \leq l \leq N \) are linearly independent, \( P_N v \) is determined uniquely. By using such notations, scheme (1.6) is equivalent to
\[
\begin{align*}
\{ P_N L u_N(x,t) &= P_N f(x,t), \quad t > 0, \\
u_N(x,0) &= P_N U_0(x).
\end{align*}
\] (1.7)

We next deduce the pseudospectral methods. In this case, \( \phi_l(x) \) are the same as in spectral methods and \( u_B(x,t) \equiv 0 \). We choose suitable collocation points \( x^{(j)}, 0 \leq j \leq N \), such that
\[
\begin{vmatrix}
\phi_0(x^{(0)}) & \cdots & \phi_0(x^{(N)}) \\
\vdots & \ddots & \vdots \\
\phi_N(x^{(0)}) & \cdots & \phi_N(x^{(N)})
\end{vmatrix} \neq 0.
\] (1.8)
The weight functions are
\[
\psi_j(x) = \delta(x - x^{(j)}), \quad 0 \leq j \leq N
\]
where \( \delta(x) \) is the Dirac delta function. Then (1.5) stands for
\[
\begin{align*}
\{ L u_N(x^{(j)},t) &= f(x^{(j)},t), \quad 0 \leq j \leq N, \quad t > 0, \\
u_N(x^{(j)},0) &= U_0(x^{(j)}), \quad 0 \leq j \leq N.
\end{align*}
\] (1.9)
It is also more convenient to express scheme (1.9) by an interpolation operator \( I_N \). For any \( v \in C(\bar{\Omega}) \), the interpolant \( I_N v \in V_N \), satisfies
\[
I_N v(x^{(j)}) = v(x^{(j)}), \quad 0 \leq j \leq N.
\]
By (1.8), the interpolant \( I_N v \) is determined uniquely. Thus (1.9) is equivalent to
\[
\begin{align*}
\{ I_N L u_N(x,t) &= I_N f(x,t), \quad t > 0, \\
u_N(x,0) &= I_N U_0(x).
\end{align*}
\] (1.10)
There is another way to express pseudospectral methods. For instance, if \( x^{(j)} \) are distributed uniformly and if we define the discrete inner product as
\[
(v, w)_N = \frac{1}{N+1} \sum_{j=0}^N v(x^{(j)})w(x^{(j)}),
\]
then (1.9) and (1.10) are equivalent to
\[
\begin{align*}
\{ (L u_N(t), \phi_l)_N = (f(t), \phi_l)_N , & \quad 0 \leq l \leq N, \quad t > 0, \\
(u_N(0), \phi_l)_N = (U_0, \phi_l)_N , & \quad 0 \leq l \leq N.
\end{align*}
\] (1.11)
We now turn to the Tau methods. Suppose that \( \phi_l(x), 0 \leq l \leq N \) are orthogonal in \( L^2(\Omega) \), but do not fulfill the boundary conditions. The component \( u_B(x, t) \) in (1.4) is of the form

\[
u_B(x, t) = \sum_{i=N+1}^{N+k} u_{N,i}(t) \phi_i(x)
\]

where \( k \) is the number of independent boundary conditions. We take \( \psi_j(x) = \phi_j(x), 0 \leq l \leq N \). Then (1.5) is read as

\[
\begin{align*}
\{ (L u_N(t), \phi_i) &= (f(t), \phi_i), \quad 0 \leq l \leq N, t > 0, \\
(u_N(0), \phi_i) &= (U_0, \phi_i), \quad 0 \leq l \leq N
\}
\end{align*}
\]

(1.12)

with the \( k \) additional equations given by the boundary conditions. The Tau methods can also be described by a projection operator, namely

\[
\begin{align*}
P_N L u_N(x, t) &= P_N f(x, t), \quad t > 0, \\
u_N(x, 0) &= P_N U_0(x)
\end{align*}
\]

(1.13)

with the \( k \) additional equations on \( \partial \Omega \).

When we solve initial-boundary value problems numerically, we need to discretize the derivatives of unknown functions with respect to time \( t \). Denote by \( [T] \) the integer part of any fixed positive constant \( T \). Let \( \tau \) be the mesh size in time \( t \) and

\[R_\tau(T) = \left\{ t = k \tau \mid k = 1, 2, \ldots, \left[ \frac{T}{\tau} \right] \right\}, \quad \bar{R}_\tau(T) = R_\tau(T) \cup \{0\}.
\]

We use the following notations

\[
\begin{align*}
\dot{v}(x, t) &= \frac{1}{2}v(x, t + \tau) + \frac{1}{2}v(x, t - \tau), \\
D_\tau v(x, t) &= \frac{1}{\tau}(v(x, t + \tau) - v(x, t)), \\
\ddot{D}_\tau v(x, t) &= \frac{1}{\tau^2}(v(x, t) - v(x, t - \tau)), \\
\dot{D}_\tau v(x, t) &= \frac{1}{2\tau}(v(x, t + \tau) - v(x, t - \tau)), \\
\ddot{D}_\tau v(x, t) &= \frac{1}{\tau^2}(v(x, t + \tau) - 2v(x, t) + v(x, t - \tau)).
\end{align*}
\]

As an example, we consider the problem

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial x^2} + f, \quad 0 < x < \pi, 0 < t \leq T, \\
U(0, t) &= U(\pi, t) = 0, \quad 0 < t \leq T, \\
U(x, 0) &= U_0(x), \quad 0 \leq x \leq \pi
\end{align*}
\]

(1.14)

where \( f(x, t) \) and \( U_0(x) \) are given continuous functions, and \( U_0(x) \) vanishes at \( x = 0, \pi \). We take \( \phi_l(x) = \sin lx \) in (1.6), and let

\[
u_N(x, t) = \sum_{i=1}^{N} u_{N,i}(t) \sin lx.
\]
A spectral scheme of Crank-Nicolson type for (1.14) is

\[
\begin{align*}
(D_{x} u_{N}(t), \phi_{i}) &= \left( \frac{1}{2} \frac{\partial^{2} u_{N}}{\partial x^{2}} (t + \tau) + \frac{1}{2} \frac{\partial^{2} u_{N}}{\partial x^{2}} (t) \right) + f(x, t + \frac{\tau}{2}), \\
(u_{N}(0), \phi_{i}) &= (U_{0}, \phi_{i}),
\end{align*}
\]

0 \leq l \leq N, t \in \tilde{R}_{\tau}(T),

(1.15)

Further, by the orthogonality of trigonometric functions, we obtain that

\[
\begin{align*}
D_{x} u_{N,l}(t) &= -\frac{\tau^{2}}{2} u_{N,l}(t + \tau) - \frac{\tau^{2}}{2} u_{N,l}(t) + f_{l} \left( t + \frac{\tau}{2} \right), \\
\left. u_{N,l}(0) \right| &= U_{0,l},
\end{align*}
\]

0 \leq l \leq N,

where \( U_{0,l} \) and \( f_{l} \left( t + \frac{\tau}{2} \right) \) are the Fourier coefficients of \( U_{0}(x) \) and \( f \left( x, t + \frac{\tau}{2} \right) \), respectively. Finally

\[
u_{N,l}(t + \tau) = \left( 1 + \frac{\tau^{2}}{2} \right)^{-1} \left( \left( 1 - \frac{\tau^{2}}{2} \right) u_{N,l}(t) + \tau f_{l} \left( t + \frac{\tau}{2} \right) \right), \quad t \in \tilde{R}_{\tau}(T).
\]

Therefore we can evaluate the coefficients \( u_{N,l}(t), 0 \leq l \leq N, t \in \tilde{R}_{\tau}(T) \) explicitly. This is indeed another advantage of spectral methods. Conversely, if a finite difference scheme of Crank-Nicolson type is used for (1.14), then we have to solve a linear algebraic system for the values of unknown function at each time step. It is also true for any finite element scheme of Crank-Nicolson type. On the other hand, the matrices occurring in finite element methods may weaken the stability of computation.

Even though the spectral methods have many advantages, they were not used widely for a long time. The main reason is the expensive cost of computational time. However the discovery of the Fast Fourier Transformation (FFT), see Cooley and Tukey (1965), removed this obstacle. Let \( N \) be the number of terms in one-dimensional Fourier expansion. Then FFT reduces the total number of operations from \( O(N^{2}) \) to \( O(N \log_{2} N) \). Especially, FFT saves a lot of work for multi-dimensional problems, since trial functions in several dimensions are the products of several trial functions in one-dimension. As Chebyshev polynomials can be changed into trigonometric polynomials by a transformation of independent variables, FFT is also available for Chebyshev spectral approximations. Furthermore, the possible parallelization of FFT makes the spectral methods more efficient.

The first serious application of the spectral methods to partial differential equations was due to Silberman (1954). However they have become practical only after Orszag (1969, 1970) and Eliasen, Machtenbauer and Rasmussen (1970) developed transform technique for evaluating convolution sums arising from quadratic non-linearities. The collocation approach was first used by Slater (1934) and Kantorovich (1934) in specific applications, and the foundation of orthogonal collocation was laid by Lanczos (1938). But the earliest application of the pseudospectral methods to partial differential equations was made by Kreiss and Oliger (1972), and Orszag (1972). Lanczos (1938) also developed the Tau methods. Gottlieb and Orszag (1977) summarized the state of art in the theory and application of spectral methods. They provided numerical analysis for linear problems. Hald (1981), Maday and Quarteroni (1981, 1982a, 1982b) considered nonlinear problems. While Guo (1981, 1985) and Kuo (1983) developed spectral methods for nonlinear problems...
with numerical analysis independently. Some new developments in this field are con-
tained in the books by Canuto, Hussaini, Quarteroni and Zang (1988), by Bernardi
and Maday (1992, 1997), and by Guo (1998). On the other hand, applications of
spectral methods in meteorology have been covered in the review by Jarraud and
Baede (1985), and in the book by Haltiner and Williams (1980).
Orthogonal Approximations in Sobolev Spaces

The remarkable convergence of spectral methods is due to the rapid convergence of expansions in series of orthogonal systems of smooth functions. We present a summary of the relevant theory in this lecture.

2.1. Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Denote by $\Omega$ a bounded open domain in $\mathbb{R}^n$ with the boundary $\partial \Omega$. If for any $x \in \partial \Omega$, there exists a system of orthogonal coordinates $y = (y_1, y_2, \ldots, y_n)$, a hypercube $U_x = \prod_{q=1}^{n} [-a_q, a_q]$ and a Lipschitz continuous mapping $\Phi_x$ from $\prod_{q=1}^{n-1} [-a_q, a_q]$ into $[-\frac{1}{2}a_n, \frac{1}{2}a_n]$ such that

\[
\Omega \cap U_x = \{ y \in U_x, y_n > \Phi_x (y_1, \ldots, y_{n-1}) \},
\]

\[
\partial \Omega \cap U_x = \{ y \in U_x, y_n = \Phi_x (y_1, \ldots, y_{n-1}) \},
\]

then we say that $\partial \Omega$ is Lipschitz continuous. Further, let $m$ be a non-negative integer. A relatively open part $\Gamma \in \partial \Omega$ is said to be of class $C^{m,1}$, if for any $x \in \Gamma$, we can choose such a mapping $\Phi_x$ that its derivatives up to the order $m$ are Lipschitz continuous. We assume that $\partial \Omega$ is at least Lipschitz continuous, and $\overline{\Omega} = \Omega \cup \partial \Omega$.

Let $D(\Omega)$ be the space of infinitely differentiable functions with compact supports in $\Omega$, and $C^\infty (\overline{\Omega})$ be the space of infinitely differentiable functions on $\overline{\Omega}$. The dual space $D'(\Omega)$ of $D(\Omega)$ is the space of distributions on $\Omega$. The notation $\partial_q$ stands for $\frac{\partial}{\partial x_q}$, and $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)$. Let $k_q, 1 \leq q \leq n$ be non-negative integers. For any $n$-tuple $k = (k_1, k_2, \ldots, k_n), |k| = k_1 + k_2 + \cdots + k_n$. The symbol $\partial_x^k = \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n}$.

When $n = 1, \partial_x = \partial_1$. In addition, $\partial_l = \frac{\partial}{\partial x_l}$ etc..

We now introduce the Sobolev spaces. For any real number $p, 1 \leq p \leq \infty$, let

\[L^p(\Omega) = \{ v \mid v \text{ is measurable and } ||v||_{L^p(\Omega)} < \infty \}\]

with the norm

\[ ||v||_{L^p(\Omega)} = \left( \int_{\Omega} |v(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,\]

\[ ||v||_{L^p(\Omega)} = \text{ess sup}_{x \in \Omega} |v(x)|, \quad p = \infty. \]
$L^p(\Omega)$ is a Banach space. In particular, $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(v, w)_{L^2(\Omega)} = \int_{\Omega} v(x)\overline{w}(x) \, dx, \quad \forall \, v, w \in L^2(\Omega)$$

where $\overline{w}$ is the conjugate of $w$, if $w$ is a complex-valued function.

Next, let $m$ be any non-negative integer, and set

$$W^{m,p}(\Omega) = \{ v \mid \partial_x^2 v \in L^p(\Omega), \ |k| \leq m \}.$$

It is provided with the following semi-norm and norm,

$$|v|_{W^m,p(\Omega)} = \left( \sum_{|k|=m} ||\partial^k v||_{L^p(\Omega)}^p \right)^{1/p},$$

$$\|v\|_{W^m,p(\Omega)} = \left( \sum_{k=0}^m \|\partial_x^k v\|_{L^p(\Omega)}^p \right)^{1/p}.$$

$W^{m,p}(\Omega)$ is a Banach space. In particular, $W^{m,2}(\Omega) = H^m(\Omega)$. It is a Hilbert space for the associated inner product

$$(v, w)_{H^m(\Omega)} = \sum_{|k| \leq m} (\partial_x^k v, \partial_x^k w)_{L^2(\Omega)}, \quad \forall \, v, w \in H^m(\Omega).$$

For any non-negative number $r$, we define the space $W^{r,p}(\Omega)$ by the interpolation between the spaces $W^{r,p}(\Omega)$ and $W^{r+1,p}(\Omega)$, see Adams (1975). For $r \geq 0$, let $W^{r,p}_0(\Omega)$ be the closure of $D(\Omega)$ in $W^{r,p}(\Omega)$. Then for $r < 0$ and $1 < p < \infty$, define the Sobolev space $W^{r,p}(\Omega)$ by

$$W^{r,p}(\Omega) = [W^{-r,q}_0(\Omega)]', \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In particular, $W^{r,2}(\Omega) = H^r(\Omega)$. For $r > 0$, it is a Hilbert space for the related inner product. For $r < 0$,

$$\|v\|_{H^r(\Omega)} = \sup_{g \in H^{-r}(\Omega)} \frac{(v, g)}{|g|_{H^{-r}(\Omega)}}$$

where $(\cdot, \cdot)$ is the duality paring between $H^{-r}(\Omega)$ and $H_0^{r}(\Omega)$. For simplicity, we denote the inner product, the semi-norm and the norm of $H^m(\Omega)$ by $(\cdot, \cdot)_m, \| \cdot \|_m$ and $\| \cdot \|_m$, respectively. Moreover $(\cdot, \cdot) = (\cdot, \cdot)_0$ and $\| \cdot \| = \| \cdot \|_0$.

Now let $X_1$ and $X_2$ be two separable Hilbert spaces, $X_1$ being continuously imbedded and dense in $X_2$. For $0 < \theta < 1$, let $[X_1, X_2]_\theta$ be the interpolation between $X_1$ and $X_2$, with the index $\theta$. In this case, we have

$$\|v\|_{[X_1, X_2]_\theta} \leq \|v\|_{X_1}^{1-\theta} \|v\|_{X_2}^\theta.$$

In particular, for $0 \leq \mu \leq r$,

$$[H^r(\Omega), H^\mu(\Omega)]_\theta = H^{(1-\theta)r+\theta\mu}(\Omega).$$
Now let $\Omega = (0, 2\pi)^n$ and $D_p(\Omega)$ be the space of infinitely differentiable functions with the period $2\pi$ for all variables. Similarly we can define $L^p_\Omega(\Omega)$, $H^r_p(\Omega)$ and $W^{r,q}_p(\Omega)$. Moreover let

$$A(v) = \int_{\Omega} v(x) \, dx, \quad \forall v \in L^1(\Omega)$$

and

$$L^2_\Omega(\Omega) = \{ v \mid v \in L^2(\Omega), A(v) = 0 \}.$$ 

Finally let $c$ denote a generic positive constant depending only on the geometry of $\Omega$.

### 2.2. Fourier Approximations

We first consider the Fourier approximations. Let $n = 1$ and $\Lambda = (0, 2\pi)$. The set of functions $e^{ilx}, l = 0, \pm 1, \ldots$, is an orthogonal system in $L^2(\Lambda)$. The Fourier transformation of a function $v \in L^2(\Lambda)$ is

$$Sv = \sum_{l=-\infty}^{\infty} \hat{v}_l e^{ilx}$$

where $\hat{v}_l$ is the Fourier coefficient,

$$\hat{v}_l = \frac{1}{2\pi} \int_\Lambda v(x) e^{-ilx} \, dx, \quad l = 0, \pm 1, \ldots$$

According to Riesz Theorem, $Sv$ converges to $v$ in $L^2(\Lambda)$, and the Parseval equality holds, namely

$$\|v\|^2 = 2\pi \sum_{l=-\infty}^{\infty} |\hat{v}_l|^2.$$

Conversely for any complex sequence $\{a_l\}$ such that $\sum_{l=-\infty}^{\infty} |a_l|^2 < \infty$, there exists a unique function $v \in L^2(\Lambda)$ such that its Fourier coefficients $\hat{v}_l = a_l, l = 0, \pm 1, \ldots$. The convergence in $L^2(\Lambda)$ does not imply the pointwise convergence of $Sv$ to $v$ at all points of $\Lambda$. However $Sv$ converges to $v$ for all $x$ outside a set of zero measures in $\Lambda$. Some other results are as follows,

(i) if $v$ is continuous, periodic and of bounded variation on $\Lambda$, then $Sv$ is uniformly convergent to $v$ on $\Lambda$;

(ii) if $v$ is of bounded variation on $\Lambda$, then $Sv$ is convergent-pointwise to $\frac{1}{2}v(x^+) + \frac{1}{2}v(x^-)$ for any $x \in \Lambda$;

(iii) if $v$ is continuous and periodic, $Sv$ does not necessarily converge at every point $x \in \Lambda$.

For discussion of differentiation, let $m$ be a positive integer and $H^m_p(\Lambda)$ be the subspace of $H^m(\Lambda)$ which consists of all functions whose first $m - 1$ derivatives are periodic. In this space, it is permissible to differentiate termwise the Fourier series $m$ times, in the sense of square mean. Indeed,

$$\partial^m v(x) = \sum_{l=-\infty}^{\infty} (il)^m \hat{v}_l e^{ilx}.$$
Consequently, the norm $\|v\|_m$ in $H_p^m(\Lambda)$ is equivalent to

$$
\left( \sum_{i=-\infty}^{\infty} (1 + l^2)^m |\hat{v}_i|^2 \right)^{\frac{1}{m}}.
$$

For $r = m + \sigma$ and $0 < \sigma < 1$, by the space interpolation,

$$
H_p^r(\Lambda) = [H_p^{m+1}(\Lambda), H_p^m(\Lambda)]_{1-\sigma}
$$

and for $v \in H_p^r(\Lambda)$,

$$
\|v\|_r \leq \|v\|_{m+1}^\sigma \|v\|_m^{1-\sigma}.
$$

For $r < 0$, let $H_p^r(\Lambda) = (H_p^{-r}(\Lambda))^\prime$. Therefore, for any $r, H_p^r(\Lambda)$ has the equivalent norm

$$
\left( \sum_{i=-\infty}^{\infty} (1 + l^2)^r |\hat{v}_i|^2 \right)^{\frac{1}{r}}.
$$

Now let $N$ be any positive integer and $\tilde{V}_N$ be the set of all trigonometric polynomials of degree at most $N$, i.e.,

$$
\tilde{V}_N = \text{span} \{ e^{itx} \mid |l| \leq N \}. \quad (2.1)
$$

Denote by $V_N$ the subset of $\tilde{V}_N$ consisting of all real-valued functions. In numerical analysis of spectral methods, we need some inverse inequalities.

**Theorem 2.1.** If $1 \leq p \leq q \leq \infty$, then for any $\phi \in V_N$,

$$
\|\phi\|_{L^q} \leq \left( \frac{Np_0 + 1}{2\pi} \right)^{\frac{1}{p} - \frac{1}{q}} \|\phi\|_{L^p}
$$

where $p_0$ is the least even number greater than or equal to $p$.

**Theorem 2.2.** Let $m$ be a non-negative integer and $1 \leq p \leq \infty$. Then for any $\phi \in \tilde{V}_N$,

$$
\|\partial_m^p \phi\|_{L^p} \leq (2N)^m \|\phi\|_{L^p}.
$$

Also for any $r \geq 0$,

$$
\|\phi\|_r \leq cN^r \|\phi\|.
$$

The proof of the above two theorems can be found in Butzer and Nessel (1971).

The $L^2$-orthogonal projection $P_N : L^2(\Lambda) \to \tilde{V}_N$ is such a mapping that for any $v \in L^2(\Lambda)$,

$$
(v - P_Nv, \phi) = 0, \quad \forall \phi \in \tilde{V}_N. \quad (2.2)
$$

If $v$ is suitably smooth, then

$$
\|v - P_Nv\|_{L^\infty} \leq \sum_{|l| > N} |\hat{v}_l|.
$$
Thus the error between \( v \) and \( P_N v \) depends on how fast the Fourier coefficients decay to zero. This in turn depends on the regularity and the periodicity of \( v \). In fact, for any \( v \in C^r(\Lambda) \) and \( l \neq 0 \),

\[
2\pi \hat{v}_l = -\frac{1}{i} \left( v(2\pi^{-}) - v(0^{+}) \right) + \frac{1}{i l} \int_{\Lambda} \partial_{y} v(x) e^{-ix \cdot y} \, dx
\]

and so \( \hat{v}_l = \mathcal{O} \left( \frac{1}{l} \right) \). Because the last integral is the Fourier coefficient of \( \partial_{y} v \), we conclude that if \( v \in C^2(\Lambda) \) and \( v(2\pi^{-}) = v(0^{+}) \), then \( \hat{v}_l = \mathcal{O} \left( \frac{1}{l} \right) \). The repetition of the above procedure tells us that if \( v \in C^m(\Lambda) \) and \( \partial_{y}^k v \) are periodic for all \( k \leq m - 2 \), then \( \hat{v}_l = \mathcal{O} \left( l^{-m} \right) \). But most recent work is associated with global approximations, started by Pascaik (1980) and other authors.

**Theorem 2.3.** If \( r \geq 0 \) and \( \mu \leq r \), then for any \( v \in H^r_{\mu}(\Lambda) \),

\[
\| v - P_N v \|_{\mu} \leq c N^{\mu-r} \| v \|_{r}.
\]

We have seen that \( P_N \partial_{y} v = \partial_{y} P_N v \). Hence \( P_N \) is also the best approximation of \( v \) in \( \tilde{V}_N \) for the norm \( \| \cdot \|_{m} \), \( m \) being any non-negative integer. However it is not so for the \( L^p \)-norm, \( 1 \leq p \leq \infty, p \neq 2 \). But we still have some results. For instance,

\[
\inf_{\phi \in \tilde{V}_N} \| v - \phi \|_{L^p} \leq c N^{-m} \| \partial_{y}^m v \|_{L^p}, \quad 1 \leq p \leq \infty.
\]

In particular, \( c = \frac{\pi}{2} \) for \( p = \infty \). Also, we have

\[
\| v - P_N v \|_{L^p} \leq c(1 + \sigma(p) \ln N) \inf_{\phi \in \tilde{V}_N} \| v - \phi \|_{L^p}
\]

with \( \sigma(p) = 0 \) for \( 0 < p < 1 \) and \( \sigma(p) = 1 \) for \( p = 1, \infty \).

In many applications, the numerical algorithms based on Fourier transformation can be implemented precisely. So we prefer to use discrete Fourier transformation. Let \( \Lambda_N \) be the set of interpolation points,

\[
\Lambda_N = \left\{ x^{(j)} \left| \frac{x^{(j)}}{2N+1}, \quad j = 0, \ldots, 2N \right. \right\}.
\]

(2.3)

The discrete Fourier transformation of function \( v \in C(\Lambda) \) is

\[
I_N v(x) = \sum_{|l| \leq N} \hat{v}_l e^{ix \cdot y}
\]

(2.4)

such that

\[
I_N v \left( x^{(j)} \right) = v \left( x^{(j)} \right), \quad 0 \leq j \leq 2N.
\]

We have

\[
\hat{v}_l = \frac{1}{2N+1} \sum_{j=0}^{2N} v \left( x^{(j)} \right) e^{-ix \cdot y^{(j)}}.
\]

The coefficient \( \hat{v}_l \) can be expressed also in terms of the coefficient \( \hat{v}_l \). In fact,

\[
\hat{v}_l = \sum_{p=-\infty}^{\infty} \hat{v}_{l+p(2N+1)}.
\]
The interpolation $I_N$ can be regarded as an orthogonal projection upon $\tilde{V}_N$ with respect to the discrete inner product. To do this, we define the discrete inner product and the norm as

$$(v, w)_N = \frac{2\pi}{2N + 1} \sum_{j=0}^{2N} v(x^{(j)}) \tilde{w}(x^{(j)}), \quad ||v||_N = (v, v)_N^{1/2}. \quad (2.5)$$

Then

$$(I_N v, \phi)_N = (v, \phi)_N, \quad \forall \ v \in C(\tilde{\Lambda}), \ \phi \in \tilde{V}_N.$$ 

This shows that $I_N v$ is the orthogonal projection of $v$ for the inner product $(\cdot, \cdot)_N$.

As in the case of Fourier transformation, the following results hold,

(i) if $v$ is continuous, periodic and of bounded variation on $\tilde{\Lambda}, I_N v$ tends to $v$ uniformly on $\tilde{\Lambda};$

(ii) if $v$ is of bounded variation on $\tilde{\Lambda}, I_N v$ is uniformly bounded on $\tilde{\Lambda}$ and converges to $v$ at every continuity point for $v$;

(iii) for any integer $l \neq 0$, and any positive $N$ such that $N > |l|$, let $\tilde{v}_l = \tilde{v}_l^{(N)}$ be the $l$-th Fourier coefficient of $I_N v$. If $v \in C^\infty_p(\tilde{\Lambda})$, then $|\tilde{v}_l^{(N)}|$ decays faster than $N^{-m}$, $m$ being any positive integer, uniformly in $N$. If $v$ satisfies the hypotheses for which $\tilde{v}_l = O(l^{-m})$, then $\tilde{v}_l^{(N)}$ possesses the same asymptotic behavior uniformly in $N$.

**Theorem 2.4.** If $r > \frac{1}{2}$ and $0 \leq \mu \leq r$, then for any $v \in H^r_p(\Lambda)$,

$$||v - I_N v||_\mu \leq c N^{\mu - r} |v|_r.$$ 

The following result provides the estimate in $L^\infty$-norm,

$$||v - I_N v||_{L^\infty} \leq c (\log N) N^{-r} |v|_{W^{r, \infty}}.$$ 

Finally, $\partial_x I_N v \neq I_N \partial_x v$ unlike $P_N$. But for any $v \in H^r_p(\Lambda)$ with $r > \frac{1}{2}$,

$$||I_N \partial_x v - \partial_x I_N v|| \leq c N^{1-r} ||v||_r.$$ 

**2.3. Orthogonal Systems of Polynomials In A Finite Interval**

Let $\omega(x)$ be a non-negative, continuous and integrable real-valued function in the interval $\Lambda = (-1, 1)$. The associated weighted space of real-valued functions is

$$L^p_\omega(\Lambda) = \{ v \mid ||v||_{L^p_\omega(\Lambda)} < \infty \}$$

equipped with the norm

$$||v||_{L^p_\omega(\Lambda)} = \left( \int_{\Lambda} |v(x)|^p \omega(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$ 

If $p = 2$, then $L^2_\omega(\Lambda)$ is a Hilbert space for the weighted inner product

$$(v, w)_{L^2_\omega(\Lambda)} = \int_{\Lambda} v(x) \overline{w(x)} \omega(x) \, dx, \quad \forall v, w \in L^2_\omega(\Lambda).$$
For any non-negative integer $m$, the Sobolev space $W^{m,p}_\omega(\Lambda)$ is defined as

$$W^{m,p}_\omega(\Lambda) = \{ v \mid \partial^j_x v \in L^p_\omega(\Lambda), 0 \leq k \leq m \}.$$ 

For any real $r \geq 0$, $W^{m,p}_\omega(\Lambda)$ is defined by the space interpolation with the corresponding norm. $W^{r,p}_\omega(\Lambda)$ is the closure of $\mathcal{D}(\Lambda)$ in $W^{r,p}_\omega(\Omega)$. For $r < 0$, $W^{r,p}_\omega(\Lambda) = (W^{-r,q}_0(\Omega))^\prime$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $W^{\infty,2}_\omega(\Lambda) = H^r_\omega(\Lambda)$. It is a Hilbert space for the related inner product when $r \geq 0$. While for $r < 0$,

$$\|v\|_{H^r_\omega(\Lambda)} = \sup_{\alpha \in N^d_0} \frac{\langle v, \alpha \rangle}{\|\alpha\|_{H^{-r}_\omega(\Lambda)}}.$$ 

For simplicity, we denote $(\cdot, \cdot)_{L^p_\omega(\Lambda)}, \|\cdot\|_{L^p_\omega(\Lambda)}, \|\cdot\|_{L^q_\omega(\Lambda)}, \|\cdot\|_{H^r_\omega(\Lambda)}$ and $\|\cdot\|_{W^{r,p}_\omega(\Lambda)}$ by $(\cdot, \cdot)_\omega, \|\cdot\|_\omega, \|\cdot\|_{L^p_\omega}, \|\cdot\|_{L^q_\omega}$ and $\|\cdot\|_{W^{r,p}_\omega}$, respectively. Also, $C(a, b; W^{r,p}_\omega(\Lambda))$ and $H^{\gamma}(a, b; W^{r,p}_\omega(\Lambda))$ are denoted by $C(a, b; W^{r,p}_\omega)$ and $H^{\gamma}(a, b; W^{r,p}_\omega)$ with the norms $\|\cdot\|_{C(a, b; W^{r,p}_\omega)}$ and $\|\cdot\|_{H^{\gamma}(a, b; W^{r,p}_\omega)}$, respectively, etc.,

Let $\phi_i(x), i = 0, 1, \ldots$, be an orthogonal system of algebraic polynomials with respect to the weighted inner product $(\cdot, \cdot)_\omega$. The formal series of a function $v \in L^q_\omega(\Lambda)$ in terms of $\{\phi_i\}$ is

$$Sv = \sum_{i=0}^{\infty} \hat{\phi}_i \phi_i(x)$$

with

$$\hat{\phi}_i = \frac{1}{\|\phi_i\|_\omega^2} \int_\Lambda v(x) \phi_i(x) \omega(x) dx.$$ 

$Sv$ is convergent to $v$ in $L^q_\omega(\Lambda)$. The rate of convergence depends on the choice of the weight function $\omega(x)$.

Denote by $P_N$ the set of all algebraic polynomials of degree at most $N$ in $\Lambda$. In this set, several inverse inequalities are valid, see Timan (1963) and Guo (1998).

**Theorem 2.5.** If $\phi_i \in L^q_\omega(\Lambda)$ for all $i$, and for certain positive constant $c_0$ and real number $\delta$,

$$|\phi_i|_{L^q_\omega} \leq c_0, \quad |\phi_i|_{L^q_\omega} \leq c_0^q |\phi_i|_\omega, \quad l \geq 1,$$

then for any $\phi \in P_N$ and all $1 \leq p \leq q \leq \infty$,

$$|\phi|_{L^q_\omega} \leq c_0^q \sigma^{\frac{1}{2} -\frac{1}{4}}(N) |\phi|_{L^p_\omega}$$

where $\sigma(N) = N^{2\delta-1}$ for $\delta > -\frac{1}{2}, \sigma(N) = \ln N$ for $\delta = -\frac{1}{2}$, and $\sigma(N) = 1$ for $\delta < -\frac{1}{2}$.

**Theorem 2.6.** For any $\phi \in P_N$,

$$|\partial_{x} \phi(x)| \leq N \min \left( \frac{1}{\sqrt{1 - x^2}}, N \right) |\phi|_{L^q_\omega}.$$ 

If in addition,

$$|\partial_{x} \phi|_\omega \leq cN^2 |\phi|_\omega,$$

then for all $2 \leq p \leq \infty$,

$$|\partial_{x} \phi|_{L^q_\omega} \leq cN^2 |\phi|_{L^q_\omega}.$$
The $L^2_\omega$-orthogonal projection $P_N : L^2_\omega \rightarrow \mathbb{P}_N$ is such a mapping that for any $v \in L^2_\omega(\Lambda)$,

$$(v - P_N v, \phi)_\omega = 0, \quad \forall \phi \in \mathbb{P}_N.$$  \hspace{1cm} (2.6)

There are some close relations between orthogonal systems and Gauss-type integrations.

(i) **Gauss integration.** Let $x^{(0)}_1, \ldots, x^{(N)}_N$ be the roots of the $(N+1)$-th orthogonal polynomial $\phi_{N+1}(x)$, and $\omega^{(0)}_1, \ldots, \omega^{(N)}_N$ be the solution of the linear system

\begin{equation}
\sum_{j=0}^{N} (x^{(j)}_j)^m \omega^{(j)}_j = \int_{\Lambda} x^m \omega(x) \, dx, \quad 0 \leq m \leq N. \tag{2.7}
\end{equation}

Then $\omega^{(j)}_j > 0$ for $0 \leq j \leq N$, and for any $\phi \in \mathbb{P}_{2N+1}$,

\begin{equation}
\sum_{j=0}^{N} \phi(x^{(j)}_j) \omega^{(j)}_j = \int_{\Lambda} \phi(x) \omega(x) \, dx; \tag{2.8}
\end{equation}

(ii) **Gauss-Radau integration.** Let $x^{(0)}_1, \ldots, x^{(N)}_N$ be the $N+1$ roots of the polynomial $\psi(x) = \phi_{N+1}(x) - \phi_{N+1}(-1)\phi_N'(1)\phi_N(x)$, and $\omega^{(0)}_1, \ldots, \omega^{(N)}_N$ be the solution of (2.7). Then (2.8) holds for any $\phi \in \mathbb{P}_{2N}$;

(iii) **Gauss-Lobatto integration.** Let $x^{(0)}_1, \ldots, x^{(N)}_N$ be the $N+1$ roots of the polynomial $(1 - x^2) \psi(x)$ such that $\psi(x) = \phi_{N+1}(x) + a\phi_N(x) + b\phi_{N-1}(x)$ and $\psi(-1) = \psi(1) = 0$, and $\omega^{(0)}_1, \ldots, \omega^{(N)}_N$ be the solution of (2.7). Then (2.8) is valid for any $\phi \in \mathbb{P}_{2N-1}$.

The points $x^{(j)}_j, 0 \leq j \leq N$ are called Gauss-type interpolation points. The numbers $\omega^{(j)}_j, 0 \leq j \leq N$ are called Gauss-type weights.

In pseudospectral methods, the fundamental representations of a smooth function $v$ are in terms of its values at Gauss-type interpolation points. The interpolant is given by

$$I_N v(x) = \sum_{l=0}^{N} \hat{v}_l \phi_l(x)$$  \hspace{1cm} (2.9)

such that

$$I_N v(x^{(j)}_j) = v(x^{(j)}_j), \quad 0 \leq j \leq N.$$  

We introduce the discrete inner product and the discrete norm as

$$(v, w)_{N, \omega} = \sum_{j=0}^{N} v(x^{(j)}_j) w(x^{(j)}_j) \omega^{(j)}_j, \quad ||v||_{N, \omega} = (v, v)_{N, \omega}^{1/2}. \tag{2.10}$$

The Gauss-type integrations imply that

$$(v, w)_{N, \omega} = (v, w)_{\omega}, \quad \forall \omega \in \mathbb{P}_{2N+\lambda}$$

where $\lambda = 1$ for the Gauss interpolation, $\lambda = 0$ for the Gauss-Radau interpolation, and $\lambda = -1$ for the Gauss-Lobatto integration. Obviously,

$$(I_N v, w)_{N, \omega} = (v, w)_{N, \omega} \quad \forall v, w \in C(\bar{\Lambda}).$$
Thus $I_N v$ is the orthogonal projection of $v$ upon $\mathbb{P}_N$ with respect to the inner product given in (2.10). Furthermore,

$$(\phi_l, \phi_m)_{N,\omega} = \gamma_l \delta_{l,m}, \quad \gamma_l = \sum_{j=0}^{N} \phi_l^{(j)}(x) \omega^{(j)}, \quad 0 \leq l, m \leq N,$$

where $\delta_{l,m}$ is the Kronecker function. Therefore

$$(v, \phi_l)_{N,\omega} = (I_N v, \phi_l)_{N,\omega} = \sum_{p=0}^{N} \tilde{\gamma}_l (\phi_p, \phi_l)_{N,\omega} = \gamma_l \tilde{\gamma}_l, \quad 0 \leq l \leq N.$$  

The coefficient $\tilde{\gamma}_l$ can be expressed in terms of the coefficients $\{\tilde{\gamma}_l\}$, i.e.,

$$\tilde{\gamma}_l = \gamma_l + \frac{1}{\gamma_l} \sum_{p>N} (\phi_p, \phi_l)_{N,\omega} \tilde{\gamma}_p.$$  

There are some relations between $\|\cdot\|_\omega$ and $\|\cdot\|_{N,\omega}$, and between $(\cdot, \cdot)_\omega$ and $(\cdot, \cdot)_{N,\omega}$. In fact, for the Gauss integration and the Gauss-Radau integration, we have that

$$\|\phi\|_{N,\omega} = \|\phi\|_\omega, \quad \forall \phi \in \mathbb{P}_N$$

and

$$|(v, \phi)_{\omega} - (v, \phi)_{N,\omega}| \leq \|v - I_N\|_\omega \|\phi\|_\omega, \quad \forall \phi \in \mathbb{P}_N. \quad (2.11)$$

For the Gauss-Lobatto integration, $\|\phi\|_{N,\omega} \neq \|\phi\|_\omega$ usually. But for mostly used orthogonal systems in $\Lambda$, they are equivalent, namely, for certain positive constants $c_1$ and $c_2$,

$$c_1 \|\phi\|_\omega \leq \|\phi\|_{N,\omega} \leq c_2 \|\phi\|_\omega.$$  

In this case, for any $\phi \in \mathbb{P}_N$,

$$|(v, \phi)_{\omega} - (v, \phi)_{N,\omega}| \leq c (2\|v - P_{N-1} v\|_\omega + \|v - I_N v\|_\omega) \|\phi\|_\omega. \quad (2.12)$$

2.4. Legendre Approximations

Let $\Lambda = (-1,1)$ and $\omega(x) \equiv 1$. The Legendre polynomial of degree $l$ is

$$L_l(x) = \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l} (1 - x^2)^l.$$  

It is the $l$-th eigenfunction of the singular Sturm-Liouville problem

$$\partial_x \left( (1 - x^2) \partial_x v \right) + \lambda v = 0, \quad x \in \Lambda,$$

related to the $l$-th eigenvalue $\lambda_l = l(l + 1)$. Clearly $L_0(x) = 1, L_1(x) = x$, and they satisfy the recurrence relations

$$L_{l+1}(x) = \frac{2l+1}{l+1} x L_l(x) - \frac{l}{l+1} L_{l-1}(x), \quad l \geq 1,$$

$$(2l+1)L_l(x) = \partial_x L_{l+1}(x) - \partial_x L_{l-1}(x), \quad l \geq 1.$$
It can be checked that $L_l(1) = 1$, $L_l(-1) = (-1)^l$, $\partial_x L_l(1) = \frac{1}{2} l (l + 1)$ and $\partial_x L_l(-1) = (-1)^{l+1} \frac{l}{2} (l + 1)$. Moreover,

$$ |L_l(x)| \leq 1, \quad |\partial_x L_l(x)| \leq \frac{1}{2} l (l + 1), \quad x \in \Lambda. $$

The set of Legendre polynomials is the $L^2$-orthogonal system in $\Lambda$, i.e.,

$$ \int_\Lambda L_l(x) L_m(x) \, dx = \left( l + \frac{1}{2} \right)^{-1} \delta_{l,m}. $$

By integrating by parts, we deduce that

$$ \int_\Lambda \left( \partial_x L_l(x) \right)^2 \, dx = l(l + 1). $$

The Legendre expansion of a function $v \in L^2(\Lambda)$ is

$$ v(x) = \sum_{l=0}^{\infty} \hat{v}_l L_l(x) $$

with

$$ \hat{v}_l = \left( l + \frac{1}{2} \right) \int_\Lambda v(x) L_l(x) \, dx. $$

We now consider the differentiation. Let $\hat{v}_l^{(1)}$ be the coefficients of Legendre expansion of $\partial_x v$, $l = 0, 1, \ldots$. We have that for any $v \in H^1(\Lambda)$,

$$ \hat{v}_l^{(1)} = (2l + 1) \sum_{p=0}^{\infty} \hat{v}_p. $$

We turn to the inverse inequalities in the space $\mathbb{P}_N$, see Timan (1963).

**Theorem 2.7.** For any $\phi \in \mathbb{P}_N$ and $1 \leq p \leq q \leq \infty$,

$$ ||\phi||_{L^p} \leq (p + 1) N^2 \frac{1}{p-q} ||\phi||_{L^q}. $$

**Theorem 2.8.** Let $m$ be a non-negative integer and $2 \leq p \leq \infty$. Then for any $\phi \in \mathbb{P}_N$,

$$ ||\partial_x^m \phi||_{L^p} \leq c N^{2m} ||\phi||_{L^p}. $$

Also for any $r \geq 0$,

$$ ||\phi||_{r} \leq c N^{2r} ||\phi||. $$

The $L^2$-orthogonal projection $P_N : L^2(\Lambda) \to \mathbb{P}_N$ is such a mapping that for any $v \in L^2(\Lambda)$,

$$ (v - P_N v, \phi) = 0, \quad \forall \phi \in \mathbb{P}_N. \quad (2.13) $$

We now estimate the difference between $v$ and $P_N v$. 

**Theorem 2.9.** For any \( v \in H^r(\Lambda) \), \( r \geq 0 \) and \( \mu \leq r \),

\[
\|v - P_N v\|_\mu \leq c N^{\sigma(\mu, r)} \|v\|_r
\]

where

\[
\sigma(\mu, r) = \begin{cases} 
2\mu - r - \frac{1}{2}, & \text{for } \mu \geq 1, \\
\mu - r, & \text{for } 0 \leq \mu \leq 1, \\
\mu - r, & \text{for } \mu < 0.
\end{cases}
\]  

(2.14)

The result of Theorem 2.9 is provided by Canuto and Quarteroni (1982), except the case \( \mu < 0 \). Other estimates are as follows,

\[
\|v - P_N v\|_{L^p} \leq c N^{\frac{1}{p} - m} V(\partial^m_x v),
\]

\[
\|v - P_N v\|_{L^{\infty}} \leq c N^{\frac{1}{2} - m} \|v\|_r
\]

where \( m \) is any positive integer and \( r \geq \frac{1}{2} \), \( V(v) \) denotes the total variation of \( v \). The best approximation error in \( L^p \)-norm, \( 2 \leq p \leq \infty \) decays as the truncation error in \( L^2 \)-norm, i.e.,

\[
\inf_{\phi \in \Pi_N} \|v - \phi\|_{L^p} \leq c N^{-r} \|v\|_{W^{r,p}}.
\]

The convergence rate in Theorem 2.9 is not optimal for \( \mu > 0 \). Let \( m \) be non-negative integer. Since \( H^m(\Lambda) \) is a Hilbert space, the best approximation to \( v \) should be the orthogonal projection of \( v \) upon \( \Pi_N \), associated with the following inner product

\[
(v, w)_m = \sum_{k=0}^{m} (\partial^k_x v, \partial^k_x w), \quad \forall \ v, w \in H^m(\Lambda).
\]

The corresponding orthogonal projection satisfies

\[
(v - P_N^m v, \phi)_m = 0, \quad \forall \ \phi \in \Pi_N.
\]  

(2.15)

Furthermore, when the spectral methods are used for partial differential equations with homogeneous boundary conditions, other kinds of projections are needed for obtaining the optimal error estimates. Let

\[
\Pi_N^{m,0} = \{ \phi \in \Pi_N \mid \partial^k_x \phi(-1) = \partial^k_x \phi(1) = 0, 0 \leq k \leq m - 1 \}
\]  

(2.16)

and \( \Pi_N^0 = \Pi_N^{1,0} \) for simplicity. Set

\[
a_m(v, w) = (\partial^m_x v, \partial^m_x w), \quad \forall \ v, w \in H^m_0(\Lambda)
\]

and \( a(v, w) = a_1(v, w) \). The commonly used inner product in \( H^m_0(\Lambda) \) is \( a_m(v, w) \). The \( H^m_0 \)-orthogonal projection \( P_N^{m,0} : H^m_0(\Lambda) \to \Pi_N^{m,0} \) is such a mapping that for any \( v \in H^m_0(\Lambda) \),

\[
a_m \left( v - P_N^{m,0} v, \phi \right) = 0, \quad \forall \ \phi \in \Pi_N^{m,0}.
\]  

(2.17)

We now estimate the difference between \( v \) and \( P_N^{m,0} v \), and the difference between \( v \) and \( P_N^{m,0} v \).
Theorem 2.10. Let $m$ be a positive integer. Then for any $v \in H^r(\Lambda)$ and $0 \leq \mu \leq m \leq r$,\[ ||v - P_N^m v||_\mu \leq cN^{\mu-r}||v||_r.\]

Theorem 2.11. Let $m$ be a positive integer. Then for any $v \in H^r(\Lambda) \cap H_0^m(\Lambda)$ and $0 \leq \mu \leq m \leq r$,\[ ||v - P_N^m v||_\mu \leq cN^{\mu-r}||v||_r.\]

The proof of Theorem 2.10 and Theorem 2.11 can be found in Bernardi and Maday (1997), and Guo (1998).

We now turn to the discrete Legendre approximation. There are three kinds of Gauss-type interpolations.

(i) **Legendre-Gauss interpolation.** In this case, $x^{(j)}$ are the $N+1$ roots of $L_{N+1}(x)$, and\[ \omega^{(j)} = \frac{2}{\left(1 - (x^{(j)})^2\right) \left(\partial_x L_{N+1}(x^{(j)})\right)^2}, \quad 0 \leq j \leq N.\]

(ii) **Legendre-Gauss-Radau interpolation.** In this case, $x^{(j)}$ are the $N+1$ roots of $L_N(x) + L_{N+1}(x)$, and\[ \omega^{(0)} = \frac{2}{(N+1)^2}, \quad \omega^{(j)} = \frac{1}{(N+1)^2} \frac{1 - x^{(j)}}{\left(L_N(x^{(j)})\right)^2}, \quad 1 \leq j \leq N.\]

(iii) **Legendre-Gauss-Lobatto interpolation.** In this case, $x^{(0)} = -1, x^{(N)} = 1, x^{(j)}, 1 \leq j \leq N-1$ are the roots of $\partial_x L_N(x)$, and\[ \omega^{(j)} = \frac{2}{N(N+1)} \frac{1}{\left(L_N(x^{(j)})\right)^2}, \quad 0 \leq j \leq N.\]

The Legendre interpolant of a function $v \in C(\bar{\Lambda})$ is\[ I_N v(x) = \sum_{i=0}^{N} \tilde{v}_i L_i(x) \quad (2.18)\]

where\[ \tilde{v}_i = \frac{1}{\gamma_i} \sum_{i=0}^{N} v(x^{(j)}) L_i \left(x^{(j)}\right) \omega^{(j)} , \]

$\gamma_l = \left(l + \frac{1}{2}\right)^{-1}$ for $l < N$, $\gamma_N = \left(N + \frac{1}{2}\right)^{-1}$ for the Legendre-Gauss interpolation and the Legendre-Gauss-Radau interpolation, and $\gamma_N = \frac{2}{N}$ for the Legendre-Gauss-Lobatto interpolation. Consequently in the third case,\[ ||\phi|| \leq ||\phi||_{X,\omega} \leq \sqrt{2 + \frac{1}{N}} ||\phi||, \quad \forall \phi \in P_N.\]

Usually $\partial_x I_N v \neq I_N \partial_x v$. We have the following result.
THEOREM 2.12. For any \( v \in H^r(\Lambda), r > \frac{1}{2} \) and \( 0 \leq \mu \leq r, \)
\[
\|v - I_Nv\|_{\mu} \leq cN^{2\mu - r + \frac{1}{2}}\|v\|_{r}.
\]

Canuto and Quarteroni (1982) first proved Theorem 2.12. It could be improved in some special cases. Bernardi and Maday (1992) showed that for the Legendre-Gauss interpolation and any \( v \in H^r(\Lambda), r > \frac{1}{2}, \)
\[
\|v - I_Nv\|_{\mu} \leq cN^{\mu - r}\|v\|_{r},
\]
and for the Legendre-Gauss-Lobatto interpolation and any \( v \in H^r(\Lambda), 0 \leq \mu \leq 1, \mu < 2r - 1, \)
\[
\|v - I_Nv\|_{\mu} \leq cN^{\mu - r}\|v\|_{r}.
\]
Bernardi and Maday (1997) also showed that for the Legendre-Gauss-Lobatto interpolation and any positive integer \( m, \)
\[
|v - I_Nv|_1 \leq cN^{1-m}\|v\|_{m}.
\]

It is noted that Theorem 2.12 and (2.12) imply that for any \( v \in H^r(\Lambda), r > \frac{1}{2} \) and \( \phi \in P_N, \)
\[
|(v, \phi) - (v, \phi_N)| \leq cN^{\lambda - r}\|v\|_{r}\|\phi\|
\]
where \( \lambda = \frac{1}{2} \) in general. But \( \lambda = 0 \) for the Legendre-Gauss interpolation and the Legendre-Gauss-Lobatto interpolation.

2.5. Chebyshev Approximations

Let \( \Lambda = (-1, 1) \) and \( \omega(x) = (1 - x^2)^{-\frac{1}{2}}. \) The Chebyshev polynomial of the first kind of degree \( l \) is
\[
T_l(x) = \cos(l \arccos x).
\]
It is the \( l \)-th eigenfunction of the singular Sturm-Liouville problem
\[
\partial_x \left( (1 - x^2)^{-\frac{1}{2}} \partial_x v \right) + \lambda (1 - x^2)^{-\frac{1}{2}} v = 0, \quad x \in \Lambda,
\]
related to the \( l \)-th eigenvalue \( \lambda_l = l^2. \) Obviously \( T_0(x) = 1, T_1(x) = x, \) and they satisfy the recurrence relations
\[
T_{l+1}(x) = 2xT_l(x) - T_{l-1}(x), \quad l \geq 1,
\]
\[
2T_l(x) = \frac{1}{l+1} \partial_x T_{l+1}(x) - \frac{1}{l-1} \partial_x T_{l-1}(x), \quad l \geq 1.
\]
It can be verified that \( T_1(1) = 1, T_l(-1) = (-1)^l, \partial_x T_l(1) = l^2 \) and \( \partial_x T_l(-1) = (-1)^l + 1 l^2. \) Moreover
\[
|T_l(x)| \leq 1, \quad |\partial_x T_l(x)| \leq l^2, \quad x \in \Lambda.
\]
The set of Chebyshev polynomials is the \( L^2 \)-orthogonal system in \( \Lambda, \) i.e.,
\[
\int_{\Lambda} T_l(x)T_m(x)\omega(x) \, dx = \frac{\pi}{2} \delta_{l,m},
\]
with \( \alpha_0 = 2 \) and \( c_l = 1 \) for \( l \geq 1 \). The Chebyshev expansion of a function \( v \in L^2_\omega(\Lambda) \) is

\[
v(x) = \sum_{l=0}^{\infty} \hat{v}_l T_l(x)
\]

with

\[
\hat{v}_l = \frac{2}{\pi c_l} \int_{\Lambda} v(x) T_l(x) \omega(x) \, dx.
\]

We consider the differentiation. Let \( \hat{v}_l^{(1)} \) be the coefficients of Chebyshev expansion of \( \partial_x v, l = 0, 1, \ldots \). Then for any \( v \in H^1_\omega(\Lambda) \),

\[
\hat{v}_l^{(1)} = \frac{2}{c_l} \sum_{p+l \text{ odd}} p \hat{v}_p.
\]

We now present the inverse inequalities.

**Theorem 2.13.** For any \( \phi \in \Pi_N \) and \( 1 \leq p \leq q \leq \infty \),

\[
\frac{\|\phi\|_{L^p_\omega}}{\|\phi\|_{L^q_\omega}} \leq \left( \frac{N p_0 + 1}{\pi} \right)^{\frac{1}{q} - \frac{1}{p}} \|\phi\|_{L^p_\omega}
\]

where \( p_0 \) is the same as in Theorem 2.1.

**Theorem 2.14.** Let \( m \) be a non-negative integer and \( 2 \leq p \leq \infty \). Then for any \( \phi \in \Pi_N \),

\[
\|\partial^m \phi\|_{L^p_\omega} \leq c N^{2m} \|\phi\|_{L^p_\omega}.
\]

Also for any \( r \geq 0 \),

\[
\|\phi\|_{r, \omega} \leq c N^{2r} \|\phi\|_{\infty}.
\]

The \( L^2_\omega \)-orthogonal projection \( P_N : L^2_\omega(\Lambda) \to \Pi_N \) is such a mapping that for any \( v \in L^2_\omega(\Lambda) \),

\[
(v - P_N v, \phi)_\omega = 0, \quad \forall \phi \in \Pi_N.
\]

**Theorem 2.15.** For any \( v \in H^r(\Lambda), r \geq 0 \) and \( \mu \leq r \),

\[
\|v - P_N v\|_{\mu, \omega} \leq c N^{\sigma(\mu, r)} \|v\|_{r, \omega}
\]

where \( \sigma(\mu, r) \) is given by (2.14).

The proof of the above theorem is due to Canuto and Quarteroni (1982), except the case \( \mu < 0 \). Another estimation is that for any \( v \in W^{r,p}(\Lambda) \) and \( 1 \leq p \leq \infty \),

\[
\|v - P_N v\|_{L^p_\omega} \leq c \sigma_N(p) N^{-r} \|v\|_{W^{r,p}}
\]

where \( \sigma_N(p) = 1 \) for \( 1 < p < \infty \) and \( \sigma_N(1) = \sigma_N(\infty) = 1 + \ln N \).

In order to define the best approximation in \( H^m_\omega(\Lambda) \), we introduce the inner product

\[
(v, w)_{m, \omega} = \sum_{k=0}^{m} (\partial^k_x v, \partial^k_x w)_\omega, \quad \forall v, w \in H^m_\omega(\Lambda).
\]
The $H^m_\omega$-orthogonal projection $P^m_N : H^m_\omega(\Lambda) \to \mathbb{P}_N$ is such a mapping that for any $v \in H^m_\omega(\Lambda)$,
\[ (v - P^m_N v, \phi)_{m,\omega} = 0, \quad \forall \phi \in \mathbb{P}_N. \tag{2.20} \]

In the numerical analysis of Chebyshev spectral methods applied to partial differential equations with homogeneous boundary conditions, we need other kinds of projections for the derivations of optimal error estimates. To do this, let
\[ a_{m,\omega}(v, w) = (\partial^m_x v, \partial^m_x w)_\omega, \]
\[ a_{m,\omega}(v, w) = (\partial^m_x v, \partial^m_x (w\omega)). \]

In particular, $a_{\omega}(v, w) = a_{1,\omega}(v, w)$ and $a_{\omega}(v, w) = a_{1,\omega}(v, w)$. The $H^1_{0,\omega}$-orthogonal projection $\tilde{P}^{1,0}_N : H^1_{0,\omega}(\Lambda) \to \mathbb{P}^0_N$ is such a mapping that for any $v \in H^1_{0,\omega}(\Lambda)$,
\[ a_{\omega} \left( v - \tilde{P}^{1,0}_N v, \phi \right) = 0, \quad \forall \phi \in \mathbb{P}^0_N. \tag{2.21} \]

The other $H^1_{0,\omega}$-orthogonal projection $P^{1,0}_N : H^1_{0,\omega}(\Lambda) \to \mathbb{P}^0_N$ is such a mapping that for any $v \in H^1_{0,\omega}(\Lambda)$,
\[ a_{\omega} \left( v - P^{1,0}_N v, \phi \right) = 0, \quad \forall \phi \in \mathbb{P}^0_N. \tag{2.22} \]

**Theorem 2.16.** For any $v \in H^r_\omega(\Lambda)$ and $0 \leq \mu \leq 1 \leq r$,
\[ \|v - P^1_N v\|_{\mu,\omega} \leq cN^{\mu-r} \|v\|_{r,\omega}. \]

**Theorem 2.17.** For any $v \in H^r_\omega(\Lambda) \cap H^1_{0,\omega}(\Lambda)$ and $r \geq 1$,
\[ \|v - \tilde{P}^{1,0}_N v\|_{1,\omega} \leq cN^{1-r} \|v\|_{r,\omega}. \]

**Theorem 2.18.** For any $v \in H^r_\omega(\Lambda) \cap H^1_{0,\omega}(\Lambda)$ and $0 \leq \mu \leq 1 \leq r$,
\[ \|v - P^{1,0}_N v\|_{\mu,\omega} \leq cN^{\mu-r} \|v\|_{r,\omega}. \]

The first proof of the above three theorems can be found in Maday and Quarteroni (1981).

For the discrete Chebyshev approximations, there are also three kinds of Gauss-type interpolations.

(i) **Chebyshev-Gauss interpolation.** In this case, $x^{(j)} = \cos \frac{\pi(2j+1)}{2(N+1)}$ and $\omega^{(j)} = \frac{N+1}{N}$ for $0 \leq j \leq N$.

(ii) **Chebyshev-Gauss-Radau interpolation.** In this case, $x^{(j)} = \cos \frac{\pi j}{2N+1}$ for $0 \leq j \leq N$, and $\omega^{(0)} = \frac{\pi}{2N+1}$, $\omega^{(j)} = \frac{\pi}{N+1}$ for $1 \leq j \leq N$.

(iii) **Chebyshev-Gauss-Lobatto interpolation.** In this case, $x^{(j)} = \cos \frac{\pi j}{N}$ for $0 \leq j \leq N$, $\omega^{(0)} = \omega^{(N)} = \frac{\pi}{2N}$ and $\omega^{(j)} = \frac{\pi}{N}$ for $1 \leq j \leq N-1$. 

The Chebyshev interpolant of a function \( v \in C(\bar{\Lambda}) \) is

\[
I_N v = \sum_{l=0}^{N} \tilde{v}_l T_l(x)
\]  

(2.23)

where

\[
\tilde{v}_l = \frac{1}{\gamma_l} \sum_{l=0}^{N} v \left( x^{(j)} \right) T_l \left( x^{(j)} \right) \omega^{(j)}
\]

\( \gamma_l = \frac{2}{\pi} \) for \( l < N; \) \( \gamma_N = \frac{2}{\pi} \) for the Chebyshev-Gauss interpolation and the Chebyshev-Gauss-Radau interpolation, and \( \gamma_N = \pi \) for the Chebyshev-Gauss-Lobatto interpolation. As a consequence, for Chebyshev-Gauss-Lobatto interpolation and any \( \phi \in \mathbb{P}_N, \)

\[
\|\phi\|_{\omega} \leq \|\phi\|_{N,\omega} \leq \sqrt{2} \|\phi\|_{\omega}.
\]

In general, \( \partial_x I_N v \neq I_N \partial_x v. \) We have the following result.

**Theorem 2.19.** For any \( v \in H_r^0(\Lambda), r > \frac{1}{2} \) and \( 0 \leq \mu \leq r, \)

\[
\|v - I_N v\|_{\mu,\omega} \leq c N^{2r-\mu} \|v\|_{r,\omega}.
\]

Theorem 2.19 is cited from Canuto and Quarteroni (1982). Another estimate is that for \( r > \frac{1}{2}, \)

\[
\|v - I_N v\|_{L^\infty} \leq c N^{\frac{1}{2} - r} \|v\|_{r,\omega}.
\]

It is also shown in Bernardi and Maday (1997) that for the Chebyshev-Gauss-Lobatto interpolation and any positive integer \( m, \)

\[
|v - I_N v|_{1,\omega} \leq c N^{1-m} \|v\|_{m,\omega}.
\]

By Theorem 2.19 and (2.12), for any \( v \in H_r^0(\Lambda), r > \frac{1}{2} \) and \( \phi \in \mathbb{P}_N, \)

\[
|(v, \phi)_\omega - (v, \phi)_{N,\omega}| \leq c N^{-r} \|v\|_{r,\omega} \|\phi\|_{\omega}.
\]

In the end of this part, we state a close relation between Legendre transformation and Chebyshev transformation. Let \( \Gamma(y) \) be the Gamma function and \( \psi(y) = \Gamma \left( y + \frac{1}{2} \right) \Gamma^{-1}(y + 1). \) Denote by \( A_N \) and \( B_N \) a pair of \((N + 1) \times (N + 1)\) matrices with the elements \( A_{N,j,k} \) and \( B_{N,j,k}, \)

\[
A_{N,j,k} = \begin{cases} 
\frac{1}{\pi} \psi \left( \frac{1}{2} \right), & \text{if } 0 = j \leq k < N + 1 \text{ and } k \text{ is even}, \\
\frac{1}{\pi} \psi \left( \frac{k-j}{2} \right) \psi \left( \frac{k+j}{2} \right), & \text{if } 0 < j \leq k < N + 1 \text{ and } j + k \text{ is even}, \\
0, & \text{otherwise},
\end{cases}
\]

\[
B_{N,j,k} = \begin{cases} 
1, & \text{if } j = k = 0, \\
\frac{1}{2\psi(j)} \int_{1}^{\frac{1}{2}((j+\frac{1}{2})(k+j-1))} \psi \left( \frac{k-j-2}{2} \right) \psi \left( \frac{k+j-1}{2} \right), & \text{if } 0 < j = k < N + 1, \\
0, & \text{otherwise}.
\end{cases}
\]
Now, suppose that \( v(x) \) has a finite Legendre expansion of the form

\[
v(\cos y) = \sum_{i=0}^{N} a_i L_i(\cos y).
\]  

(2.24)

Then it also has a finite Chebyshev expansion of the form

\[
v(\cos y) = \sum_{i=0}^{N} b_i T_i(\cos y)
\]  

(2.25)

where \( a = (a_0, \ldots, a_N) \) and \( b = (b_0, \ldots, b_N) \) are related by the equation

\[
b = A_N a.
\]

Conversely, if \( v \) is a function given by (2.25), then it may be expressed in the form of (2.24), where

\[
a = B_N b.
\]

Based on the above fact, Alpert and Rokhlin (1991) developed the Fast Legendre Transformation (FLT). For a Legendre expansion of degree \( N \), FLT produces its values at \( N + 1 \) Chebyshev points in \( \Lambda \) with a cost proportional to \( (N + 1) \ln(N + 1) \). Similarly, FLT produces the Legendre expansion of degree \( N \) from the values of \( v \) at \( N + 1 \) Chebyshev points. The cost of this algorithm is roughly three times that of FFT of length \( N + 1 \), provided that the calculations are performed to single precision accuracy. In double precision, the ratio is approximately 5.5. FLT saves a lot of work in Legendre spectral methods and Legendre pseudospectral methods, and makes them more efficient.

2.6. Some Orthogonal Approximations In Infinite Intervals

A number of physical problems are set in unbounded domains. The first remark with spectral methods on these problems, is that polynomials are not integrable on unbounded domains. So the idea is to work with weighted Sobolev spaces where the weight must be of exponential type in order to avoid restrictions on the degree of the polynomials. We first consider the case \( \Lambda = (0, \infty) \) and \( \omega(x) = e^{-x} \). Define the spaces \( L^p_\omega(\Lambda), p \geq 1, W^{r,p}_\omega(\Lambda), H^s_\omega(\Lambda), C(a, b; W^{r,p}_\omega(\Lambda)), H^s(a, b; W^{r,p}_\omega(\Lambda)) \) and their semi-norms and norms in the same way as before. Also we set the inner product \((\cdot, \cdot)_\omega\) of \( L^2_\omega(\Lambda) \). Clearly for any bounded interval \( \Lambda^* \subset \Lambda \), the restrictions to \( \Lambda^* \) of all functions in \( H^s_\omega(\Lambda) \) belong to \( H^s_\omega(\Lambda^*) \). We quote some essential properties of \( H^s_\omega(\Lambda) \). They state that

(i) for any \( r \geq 0 \), the space \( C^\infty(\Lambda) \) is dense in \( H^s_\omega(\Lambda) \);

(ii) for any \( r \geq 0 \), the mapping: \( v(x) \rightarrow e^{-\theta} v(x) \) is an isomorphism from \( H^s_\omega(\Lambda) \) onto \( H^s(\Lambda) \);

(iii) for any \( 0 \leq \mu < r \) and \( 0 < \theta < 1 \),

\[
[H^s_\omega(\Lambda), H^s_\omega(\Lambda)]_\theta = H^{(1-\theta)r+\theta\mu}_\omega(\Lambda);
\]
(iv) for any $r > \frac{1}{2}$, the space $H^r_\omega(\Lambda)$ is contained in $C(\overline{\Lambda})$, and for any $v \in H^r(\Lambda),
\sup_{x \geq 0} |e^{-\frac{r}{2}} v(x)| \leq c\|v\|_{r, \omega}.

For technical reasons, we also need another Sobolev space associated with a positive integer $\alpha$,

$$H^\alpha_\omega(\Lambda) = \{ v \in H^\alpha_\omega(\Lambda) \mid x^{\frac{\alpha}{2}} v \in H^\alpha_\omega(\Lambda) \}.$$  

This space is provided with the associated natural norm $\|v\|_{r, \omega, \alpha} = \|v(1 + x)^\frac{\alpha}{2}\|_{r, \omega}$.

The Laguerre polynomial of degree $l$ is

$$L_l(x) = \frac{1}{l!} e^x \partial_x^l (x^l e^{-x}).$$

It is the $l$-th eigenfunction of the singular Sturm-Liouville problem

$$\partial_x (x e^{-x} \partial_x v(x)) + \lambda e^{-x} v(x) = 0,$$

related to the $l$-th eigenvalue $\lambda_l = l$. Clearly $L_0(x) = 1, L_1(x) = 1 - x$, and they satisfy the recurrence relations

$$(l + 1) L_{l+1}(x) = (2l + 1 - x) L_l(x) - (l - 1) L_{l-1}(x), \quad l \geq 1,$$

$$L_l(x) = \partial_x L_{l-1}(x) - \partial_x L_{l+1}(x), \quad l \geq 0.$$ 

It can be checked that $L_l(0) = 1, \partial_x L_l(0) = l$ for $l \geq 1$, and

$$|L_l(x)| \leq e^x, \quad x \in \Lambda.$$ 

The set of Laguerre polynomials is the $L^2_\omega$-orthogonal system on the half line $\Lambda$, i.e.,

$$\int_\Lambda L_l(x) L_m(x) \omega(x) \, dx = \delta_{l,m}.$$  

By integrating by parts, we deduce that

$$\int_\Lambda \partial_x L_l(x) \partial_x L_m(x) x \omega(x) \, dx = l \delta_{l,m}.$$ 

The Laguerre expansion of a function $v \in L^2_\omega(\Lambda)$ is

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l L_l(x)$$

with

$$\hat{v}_l = \int_\Lambda v(x) L_l(x) \omega(x) \, dx.$$ 

Now let $\mathcal{P}_N$ be the space of restrictions to $\Lambda$ of polynomials of degree at most $N$. Several inverse inequalities exist in this space, see Bernardi and Maday (1997), and Guo (1998).
**Theorem 2.20.** For any $\phi \in \mathbb{P}_N$ and $1 \leq p \leq q \leq \infty$,

$$
\|\phi\|_{L^p} \leq c N^{\frac{q-p}{p}} \|\phi\|_{L^q}.
$$

**Theorem 2.21.** For any $\phi \in \mathbb{P}_N$,

$$
\|\partial_x \phi\|_\omega \leq c N \|\phi\|_\omega.
$$

The $L^2_\omega$-orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathbb{P}_N$ is such a mapping that for any $v \in L^2_\omega(\Lambda)$,

$$
(v - P_N v, \phi)_\omega = 0, \quad \forall \phi \in \mathbb{P}_N.
$$

**Theorem 2.22.** For any $v \in H^r(\Lambda, \alpha)$ and $0 \leq \mu \leq r$,

$$
\|v - P_N v\|_{\mu, \omega} \leq c N^{\mu - \frac{r}{2}} \|v\|_{r, \omega, \alpha}
$$

where $\alpha$ is the largest integer such that $\alpha < r + 1$.

Theorem 2.22 was given by Maday, Pernaud-Thomas and Vandeveen (1985).

We now consider the case $\Lambda = (-\infty, \infty)$ and $\omega(x) = e^{-x^2}$. We define $L^p_\omega(\Lambda)$, $p \geq 1$, $H^r(\Lambda, \alpha)$ and their norms in the same way as before. The Hermite polynomial of degree $l$ is

$$
H_l(x) = (-1)^l e^{x^2} \partial_x^l \left( e^{-x^2} \right).
$$

It is the $l$-th eigenfunction of the singular Sturm-Liouville problem

$$
\partial_x \left( e^{-x^2} \partial_x v(x) \right) + \lambda e^{-x^2} v(x) = 0,
$$

related to the $l$-th eigenvalue $\lambda_l = 2l$. Clearly $H_0(x) = 1, H_1(x) = 2x$, and they satisfy the recurrence relations

$$
H_{l+1}(x) - 2x H_l(x) + 2l H_{l-1}(x) = 0, \quad l \geq 1,
$$

$$
\partial_x H_l(x) = 2l H_{l-1}(x), \quad l \geq 1.
$$

It can be checked that

$$
|H_l(x)| \leq c_2 2^{l+1} (l!)^\frac{1}{2} x^l, \quad x \in \Lambda,
$$

where $c_2 \sim 1.086435$. The set of Hermite polynomials is the $L^2_\omega$-orthogonal system on the whole line $\Lambda$, i.e.,

$$
\int_\Lambda H_l(x) H_m(x) \omega(x) \, dx = 2^l l! \sqrt{\pi} \delta_{l,m}
$$

from which,

$$
\int_\Lambda \partial_x H_l(x) \partial_x H_m(x) \omega(x) \, dx = 2^{l+1} (l+1)! \sqrt{\pi} \delta_{l,m}.
$$

The Hermite expansion of a function $v \in L^2(\Lambda)$ is

$$
v(x) = \sum_{l=0}^{\infty} \hat{v}_l H_l(x),
$$
with
\[ \hat{v}_l = \frac{1}{2^{l+1} \sqrt{\pi}} \int_{\Lambda} v(x) H_l(x) \omega(x) \, dx. \]

Now let \( \mathbb{P}_N \) be the space of polynomials of degree at most \( N \). Several inverse inequalities exist in this space, see Nessel and Wilmes (1976), and Guo (1998).

**Theorem 2.23.** For any \( \phi \in \mathbb{P}_N \) and \( 1 \leq p \leq q \leq \infty \),
\[ \|\phi\|_{L^q_p} \leq c N^{\frac{q}{p} - \frac{1}{p}} \|\phi\|_{L^p_N}. \]

**Theorem 2.24.** For any \( \phi \in \mathbb{P}_N \),
\[ \|\partial_x \phi\|_{\infty} \leq c \sqrt{N} \|\phi\|_{\infty}. \]

The \( L^2_\omega \)-orthogonal projection \( P_N : L^2_\omega(\Lambda) \to \mathbb{P}_N \) is such a mapping that for any \( \psi \in L^2_\omega(\Lambda) \),
\[ (\psi - P_N \psi, \phi) = 0, \quad \forall \phi \in \mathbb{P}_N. \]

Guo (1998) proved the following result.

**Theorem 2.25.** For any \( \psi \in H^\mu_\omega(\Lambda) \) and \( 0 \leq \mu \leq r \),
\[ \|\psi - P_N \psi\|_{\mu, \infty} \leq c N^{\frac{1}{2} - \frac{\mu}{r}} \|\psi\|_{r, \infty}. \]

For the application of Hermite approximation, we refer to Funaro and Kavian (1990).

So far, we have considered the spectral approximations based on algebraic polynomials. However we can also use other kinds of orthogonal systems. For instance, let \( \Lambda = (-\infty, \infty), \omega(x) = (x^2 + 1)^{-1} \) and
\[ R_l(x) = \cos(l \arccot x), \quad l = 0, 1, \ldots. \]

The set of \( \{R_l(x)\} \) is an orthogonal system, i.e.,
\[ \int_{\Lambda} R_l(x) R_m(x) \omega(x) \, dx = \frac{1}{2} c_l c_m \delta_{l,m} \]
where \( c_0 = 2 \) and \( c_l = 1 \) for \( l \geq 1 \). The expansion of a function \( \psi \in L^2_\omega(\Lambda) \) is
\[ \psi(x) = \sum_{l=0}^{\infty} \hat{\psi}_l(x) R_l(x) \]
with
\[ \hat{\psi}_l(x) = \frac{2}{\pi c_l} \int_{\Lambda} \psi(x) R_l(x) \omega(x) \, dx. \]

Let
\[ \mathbb{R}_N = \text{span} \{R_l(x) \mid 0 \leq l \leq N\}. \]

The \( L^2_\omega \)-orthogonal projection \( P_N : L^2_\omega(\Lambda) \to \mathbb{R}_N \) is such a mapping that for any \( \psi \in L^2_\omega(\Lambda) \),
\[ (\psi - P_N \psi, \phi) = 0, \quad \forall \phi \in \mathbb{R}_N. \]
2.7. Filterings And Recovering The Spectral Accuracy

Spectral methods have the convergence rate of “infinite order”, also called the spectral accuracy. But this merit may be destroyed by two facts. The first is the instability of nonlinear computation. The second is the lower accuracy caused by the discontinuity of data. In order to remedy these deficiencies, various techniques have been developed, such as filterings, non-oscillatory polynomial interpolations and reconstructions of orthogonal approximations.

We first discuss the improvement of stability. As we know, pseudospectral approximations are more preferable in practice, since they are only needed to evaluate unknown functions at interpolation points and it is much easier to deal with nonlinear terms. But they are less stable than the corresponding spectral ones usually, due to aliasing interaction in calculations of nonlinear terms and derivatives, such as the term \( v \partial_x v \) in the Burgers’ equation. Kreiss and Oliger (1979) first proposed the filtering operator \( R_N \) for Fourier approximations. The simplest one is based on Cesàro mean. Let \( \Lambda = (0, 2\pi) \), \( \phi \in V_N \) and \( \hat{\phi}_l \) be the Fourier coefficients of \( \phi \), \( \hat{\phi}_{-l} = \hat{\phi}_l \). Then the filtering is given by

\[
R_N \phi(x) = \sum_{|l| \leq N} \left( 1 - \left| \frac{l}{N} \right| \right) \hat{\phi}_l e^{ilx}.
\]

If \( \hat{\phi}(x) \) has the error \( \hat{\phi}(x) \), then \( R_N I_N \left( \hat{\phi}(x) \partial_x \hat{\phi}(x) \right) \) weakens the effect of high frequency components of \( \hat{\phi}(x) \) and thus improves the stability. On the other hand, if \( v \in C(\Lambda) \), then \( R_N P_N v(x) \) converges to \( v(x) \) uniformly in \( \Lambda \), while it is not so for standard Fourier approximation. The drawback of this approach is that the convergence rate of \( R_N P_N v(x) \) in \( L^\infty \)-norm is limited to \( O\left( \frac{1}{N} \right) \), no matter how smooth the function \( v(x) \) is. Kuo (1983) proposed another filtering operator \( R_N(\alpha, \beta) \) based on a generalization of Bochner summation, i.e.,

\[
R_N(\alpha, \beta) \phi(x) = \sum_{|l| \leq N} \left( 1 - \left| \frac{l}{N} \right| \right)^\alpha \hat{\phi}_l e^{ilx}, \quad \alpha, \beta \geq 1.
\]  

(2.28)

This approach is analyzed in Ma and Guo (1986), and Guo (1998). The key point is to improve the stability and keep the same accuracy as the standard pseudospectral methods.

**Theorem 2.26.** For any \( \phi \in V_N \) and \( 0 \leq r - \mu \leq \alpha \),

\[
||R_N(\alpha, \beta) \phi - \phi)||_\mu \leq c\beta N^{\mu - r} ||\phi||_r.
\]

If in addition \( \mu \geq 0 \), then

\[
||R_N(\alpha, \beta) \phi - \phi)||_\mu \leq c\beta N^{\mu - r} ||\phi||_r.
\]

Theorem 2.3 and Theorem 2.26 imply that for \( v \in H^r(\Lambda) \) and \( 0 \leq r - \mu \leq \alpha \),

\[
||R_N(\alpha, \beta) P_N v - v)||_\mu \leq c(\beta + 1) N^{\mu - r} ||v||_r.
\]

If in addition \( r > \frac{1}{2} \), then

\[
||R_N(\alpha, \beta) I_N v - v)||_\mu \leq c(\beta + 1) N^{\mu - r} ||v||_r.
\]
For fixed value of $\alpha$, the accuracy of filtered Fourier approximation is still limited. However we can take $\alpha = \alpha(N)$ and $\alpha(N) \to \infty$ as $N \to \infty$. In this case, for any $0 \leq \mu \leq r$ and suitably large $N$;

$$
\| R_N(\alpha(N), \beta) I_N v - v \|_\mu \leq c(\beta + 1) N^{\mu-r} \| v \|_r .
$$

A general filtering for $\phi \in V_N$ is of the form

$$
R_N \phi(x) = \sum_{|l| \leq N} \sigma_{N,l} \hat{\phi}_l e^{i l x}
$$

where $\sigma_{N,l} = \sigma_{N,-l}$ are real numbers, which decay smoothly from one to zero when $|l|$ goes from zero to $N$. The coefficients $\sigma_{N,l}$ could be produced by a function $\sigma(x), 0 \leq x \leq 1$ such that $\sigma(0) = 1, \sigma(1) = 0$ and $\partial_x^k \sigma(0) = \partial_x^k \sigma(1) = 0$ for $1 \leq k \leq m$. Set $\sigma_{N,l} = \sigma \left( \frac{l}{N} \right)$ in (2.29). Then for smooth function $v(x), R_N P_N v(x)$ converges to $v(x)$ pointwise in the order of $\mathcal{O} \left( \frac{1}{N^{r+\epsilon}} \right)$, see Vandenven (1991). In actual computations, we could take $\sigma(x) = e^{-\alpha x^2}$ where $\alpha$ is chosen so that $\sigma_{N,N}$ is the machine zero and $2\gamma$ is called the order of the exponential filtering.

In the early work of Gottlieb and Turkel (1980), the filterings are coupled with the mesh size $\tau$ of the time $t$ for revolutionary problems. Let $U = \{ U^{(1)}, U^{(2)}, \ldots, U^{(m)} \}$ and $A$ be a $m \times m$ matrix. They considered the problem

$$
\begin{aligned}
\partial_t U &= A \partial_x U , & x \in \Lambda, 0 < t \leq T , \\
U(x,t) &= U(x + 2\pi , t) , & 0 \leq t \leq T , \\
U(x,0) &= U_0(x), & x \in \tilde{\Lambda} .
\end{aligned}
$$

The corresponding approximate problem is

$$
U(x,t + \tau) = U(x,t - \tau) + 2\tau A \partial_x U(x,t).
$$

Further, we use discrete Fourier approximation in the space and denote by $u_N(x,t)$ the numerical solution. The last term in the above formula is evaluated as

$$
2 \sum_{|l| \leq N} \hat{u} \tau \left( \widehat{A u_N(t)} \right)_l e^{i l x} .
$$

Gottlieb and Turkel (1980) replaced the above term by

$$
2 \sum_{|l| \leq N} i \lambda_l \left( \widehat{A u_N(t)} \right)_l e^{i l x}
$$

where

$$
\lambda_l = \frac{\sin(l \alpha \tau)}{\alpha} , \quad a = \alpha \rho(A) , \quad \alpha \geq 1 ,
$$

and $\rho(A)$ is the spectral radius of matrix $A$. For hyperbolic systems, there exists a matrix $B$ such that $B A B^{-1} = D$, $D = [d_1, d_2, \ldots, d_m]$. Let $\sin(l \tau D) = [\sin l \tau d_1 , \ldots , \sin l \tau d_m]$. We can take $\lambda_l = B^{-1} \sin(l \tau D) B$. An alternative is to replace $l \tau A$ by a Padé approximation. In this case, the term $l \tau \left( \widehat{A u_N(t)} \right)$ is replaced by $\theta_l$, and $(I + \alpha^2 \tau^2 A'^2) \theta_l = l \tau \widehat{A(u_N)}$. If $A = I$, we can take $\lambda_l = \frac{l}{\alpha^2} (e^{\alpha \tau} - 1)$.
We next consider the recovering spectral accuracy. The usual Fourier expansion of a function with discontinuity converges slowly. For example, consider a sawtooth-like function

\[ f(x, x', A) = \begin{cases} 
  -Ax, & \text{for } x \leq x', \\
  2\pi A - Ax, & \text{for } x > x'
\end{cases} \]  

(2.30)

where \( x' \) is the location of the discontinuity and \( A = [f]_{x'} = \frac{f(x', x', A) - f(x', x', A)}{2\pi} \) is the jump of \( f(x, x', A) \) across \( x' \). Let \( v_N(x) = P_N v(x) \). Then

\[ f_N(x, x', A) = \sum_{|l| \leq N} \hat{f}_l (x', A) e^{ilx} \]

with \( \hat{f}_0 (x', A) = 2\pi A - Ax' \), and

\[ \hat{f}_l (x', A) = \frac{Ae^{-ilx'}}{il}, \quad |l| \geq 1. \]

We can see that \( f_l (x', A) \) only decays like \( O \left( \frac{1}{l} \right) \) as \( l \to \infty \). As a result, the convergence will be only of the first order, and moreover, the Gibbs oscillations near \( x' \) will be of the order of \( O(1) \). In order to get rid of them, several techniques have been proposed.

Cai, Gottlieb and Shu (1989) provided a method based on the property of sawtooth-like function (2.30). Assume that \( v(x) \) is a periodic piecewise \( C^\infty \) function with one discontinuity at \( x' \) and \( A = [v]_{x'} \). Let \( x^* \) and \( A^* \) be the approximations to \( x' \) and \( A \) respectively. Define

\[ v_N^* (x) = \sum_{|l| \leq N} \hat{v}_l e^{ilx} + \sum_{|l| > N} \frac{A^*}{il} e^{il(x-x*)}. \]  

(2.31)

Since the last sum is actually \( f(x, x^*, A^*) - f_N(x, x^*, A^*) \), we have

\[ v_N^*(x) = \sum_{|l| \leq N} \left( \hat{v}_l - \frac{A^*}{il} \right) e^{ilx} + f(x, x^*, A^*). \]

(2.31) yields an essentially nonoscillatory approximation provided that the approximations for the location \( x' \) and the jump \( A \) are quite accurate. Indeed, it is shown that

\[ V(v_N^*) \leq V(v) + c \frac{\ln N}{N} + cN \ln N |x' - x^*| + c \ln N |A - A^*|, \]

\[ \|v - v_N^*\|_{L^1} \leq c \frac{\ln N}{N^2} + cN |x' - x^*| + c |A - A^*| \]

where \( V(v) \) is the total variation of \( v \). Moreover, both \( |x' - x^*| \) and \( |A - A^*| \) are of the order of \( O \left( \frac{1}{N} \right) \), if \( x^* \) and \( A^* \) are determined by the following system

\[
\begin{align*}
\frac{A^*}{\pi(N+1)} e^{-i(N+1)x^*} &= \hat{v}_{N+1}^*, \\
\frac{A^*}{\pi(N+2)} e^{-i(N+2)x^*} &= \hat{v}_{N+2}^*.
\end{align*}
\]
Cai and Shu (1991) provided a uniform approximation by using an idea in Harten, Engquist, Osher and Chakravarthy (1987) and the filtering (2.29).

Another uniform approximation is based on the one-sided filtering. Suppose that the function \( v(x) \) is analytic over \([0, 2\pi)\), but \( v(0) \neq v(2\pi) \). Vandeven (1991) considered the two-sided filtering (2.29) with

\[
\sigma_{N,t} = 1 - \frac{(2p-1)!}{((p+1)!)^2} \int_0^\pi |y(1 - y)|^{p-1}dy, \quad p = N^\frac{1}{p}, \quad 0 < \varepsilon < 1, \tag{2.32}
\]

and proved that

\[
\max_{\delta \leq x \leq 2\pi - \delta} |v(x) - R_N P_N v(x)| \leq \frac{N^{\lambda}}{(cN^\frac{1}{p})^{N^\frac{1}{p}}}
\]

where \( \delta = N^{\varepsilon - 1} \), and \( \lambda \) is a certain constant independent of \( N \). Cai, Gottlieb and Shu (1992) defined the one-sided filtering as

\[
R_N^* \phi(x) = \sum_{|l| \leq N} \sigma_{N,t}^* \hat{\phi}_l e^{ilx}, \quad \forall \phi \in V_N
\]

where

\[
\sigma_{N,t}^* = \sigma_{N,t} \sum_{p=1}^m (-1)^{p+1} \frac{m!}{p!(m-p)!} \exp(ipMN^{-1}) \quad m = N^\frac{1}{p},
\]

and \( \sigma_{N,t} \) is given by (2.32). They proved that if \( v(x) \) is periodic, analytic in \([0, 2\pi)\), but \( v(0) \neq v(2\pi) \), then for any \( 0 < \varepsilon < \frac{1}{4} \),

\[
\max_{0 \leq x \leq 2\pi - \delta} |v(x) - R_N^* P_N v(x)| \leq \frac{N^{\lambda}}{(cN^\frac{1}{p})^{N^\frac{1}{p}}}
\]

where \( \delta = 2N^{\frac{1}{4} - \varepsilon} \). The above filtering is naturally labelled “right-sided” filtering since it can recover the spectral accuracy up to the discontinuity \( x = 0 \) from the right. The “left-sided” filtering can be constructed similarly.

Recently Gottlieb, Shu, Solomonoff and Vandeven (1992) developed a new method for recovering the spectral accuracy. The main idea of this method is to reconstruct the Fourier expansion by using Gegenbauer polynomials. The Gegenbauer polynomial of degree \( l \) with parameter \( \lambda \geq 0 \) is given by

\[
G_{l,\lambda}(x) = G_{l,\lambda} (1 - x^2)^{\frac{1}{2} - \lambda} \partial_x \left( (1 - x^2)^{l+\lambda - \frac{1}{2}} \right)
\]

where

\[
G_{l,\lambda} = \frac{(-1)^l \Gamma \left( \lambda + \frac{1}{2} \right) l^\lambda \Gamma(l + 2\lambda)}{2^l l! \Gamma(2\lambda) \Gamma(l + \lambda + \frac{1}{2})}, \quad \text{for } \lambda > 0,
\]

\[
G_{l,0} = \frac{(-1)^l \sqrt{\pi}}{2^{l-1} l! \Gamma(l + \frac{1}{2})}, \quad \text{for } l \geq 1
\]

and

\[
G_{0,0} = 1.
\]
By this standardization, \( C_{0,0}(x) = 1 \) and for \( t \geq 0 \),
\[
C_{t,0}(x) = \frac{2}{t} T_t(x), \quad C_{t,1}(x) = L_t(x).
\]
The Gegenbauer polynomials satisfy the recurrence relations
\[
\partial_x \left( (1 - x^2)^{-\lambda} \frac{\lambda - \frac{1}{2}}{2} C_{t,\lambda}(x) \right) = \frac{G_{t,\lambda}}{G_{t+1,\lambda-1}} (1 - x^2)^{-\lambda} \frac{\lambda - \frac{1}{2}}{2} C_{t+1,\lambda-1}(x)
\]
and
\[
2(l + \lambda) C_{l,\lambda}(x) = \partial_x C_{l+1,\lambda}(x) - \partial_x C_{l-1,\lambda}(x).
\]
It is shown that \( C_{l,\lambda}(1) = \Gamma(l+2\lambda) / \Gamma(2\lambda) \) and \( |C_{l,\lambda}(x)| \leq C_{l,\lambda}(1) \) for \( |x| \leq 1 \). Let \( \Lambda = (-1, 1) \) and \( \omega_{\lambda}(x) = (1 - x^2)^{-\lambda - \frac{1}{2}} \). The set of Gegenbauer polynomials is the \( L^2_{\omega_{\lambda}} \)-orthogonal system in \( \Lambda \), i.e.,
\[
\int_\Lambda C_{l,\lambda}(x) C_{m,\lambda}(x) \omega_{\lambda}(x) \, dx = h_{l,\lambda} \delta_{l,m}
\]
where for \( \lambda > 0 \),
\[
h_{l,\lambda} = \frac{\sqrt{\pi} C_{l,\lambda}(1) \Gamma \left( \lambda + \frac{1}{2} \right)}{(l + \lambda) \Gamma(\lambda)}.
\]
The Gegenbauer expansion of a function \( v \in L^2_{\omega_{\lambda}}(\Lambda) \) is
\[
v(x) = \sum_{l=0}^{\infty} \hat{v}_{l,\lambda} C_{l,\lambda}(x)
\]
with the coefficients
\[
\hat{v}_{l,\lambda} = \frac{1}{h_{l,\lambda}} \int_\Lambda v(x) C_{l,\lambda}(x) \omega_{\lambda}(x) \, dx, \quad l = 0, 1, \ldots.
\]
The corresponding orthogonal projection is
\[
v_{N,\lambda}(x) = \sum_{l=0}^{N} \hat{v}_{l,\lambda} C_{l,\lambda}(x). \tag{2.33}
\]
Now, we consider an analytic, but non-periodic function \( v(x) \) in \( \bar{\Lambda} \). If it is extended periodically with the period 2, then \( v(x) \) has a discontinuity at \( x = \pm 1 \). The Fourier coefficients of \( v(x) \) are defined as
\[
\hat{v}_l = \frac{1}{2} \int_\Lambda v(x) e^{-i \pi x} \, dx.
\]
The traditional truncated Fourier sum is
\[
v_N(x) = \sum_{|l| \leq N} \hat{v}_l e^{i \pi x}.
\]
Let $J_\lambda(x)$ be the Bessel function and $\hat{w}_{l,\lambda}$ be the coefficients of Gegenbauer expansion of $v_N(x)$. Then they can be expressed as

$$\hat{w}_{l,\lambda} = \delta_{l,0} \hat{h}_0 + i^l (l + \lambda) \Gamma(\lambda) \sum_{0 \leq |p| \leq N} J_{l+\lambda}(p\pi) \left( \frac{2}{p\pi} \right)^\lambda \hat{v}_p.$$ 

The reconstructed expansion of $v(x)$ is

$$v^*_N(x) = \sum_{l=0}^M \hat{w}_{l,\lambda} C_{l,\lambda}(x). \quad (2.34)$$

Assume that there is a positive constant $c(\rho)$ depending only on $\rho \geq 1$ such that

$$\max_{|x| \leq 1} \left| \frac{\partial^k v(x)}{\partial x^k} \right| \leq c(\rho) \frac{k!}{\rho^k}. \quad (2.35)$$

This is a standard assumption for analytic functions. Gottlieb, Shu, Solomonoff and Vandeven (1992) proved that if (2.35) holds, $\lambda = M = \beta N$ and $\beta < \frac{2}{\pi} \pi e$, then

$$\max_{|x| \leq 1} \left| v(x) - v^*_N(x) \right| \leq b_1 N^2 q_1^N + b_2 q_2^N$$

where $b_1$ and $b_2$ are certain positive constants related to $\max_{0 \leq l \leq \infty} |\hat{v}_l|$, and

$$q_1 = \left( \frac{27\beta}{2\pi e} \right)^\beta < 1, \quad q_2 = \left( \frac{27}{32\rho} \right)^\beta < 1.$$ 

The meanings of $q_1$ and $q_2$ are explained in their paper. (2.34) gives a uniform approximation to $v(x)$, including the discontinuity, and recovers the spectral accuracy. We also refer to work in Gottlieb and Shu (1994).

All of the methods in the previous paragraphs are also available for Legendre approximation and Chebyshev approximation. For instance, let $v \in \mathbb{P}_N$ and $\hat{v}_l$ be the Chebyshev coefficients of $v(x), 0 \leq l \leq N$. Guo and Li (1993) offered a filtering. Ma and Guo (1994) considered $R_N(\alpha,\beta)$ and $R^0_N(\alpha,\beta)$ for $\phi \in \mathbb{P}_N$, defined by

$$R_N(\alpha,\beta) \phi(x) = \sum_{l=0}^N \left( 1 - \left| \frac{l}{N} \right|^{\alpha} \right)^\beta \hat{\phi} T_l(x), \quad \alpha, \beta \geq 1, \quad (2.36)$$

$$R^0_N(\alpha,\beta) \phi(x) = \sum_{l=0}^{N-2} \left( 1 - \left| \frac{l}{N} \right|^{\alpha} \right)^\beta \hat{\phi} T_l(x) + a_{N-1} T_{N-1}(x) + a_N T_N(x), \quad \alpha, \beta \geq 1, \quad (2.37)$$

where the constants $a_{N-1}$ and $a_N$ are determined by

$$R^0_N(\alpha,\beta) \phi(-1) = \phi(-1), \quad R^0_N(\alpha,\beta) \phi(1) = \phi(1).$$

Let $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$. Ma and Guo (1994) proved the following results.
Theorem 2.27. For any $\phi \in \mathcal{P}_N$ and $0 \leq r \leq \alpha$, 
$$
\| R_N(\alpha, \beta) \phi - \phi \|_\omega \leq c \beta N^{-r} \| \phi \|_\omega.
$$

Theorem 2.28. For any $\phi \in \mathcal{P}_N$, 
$$
|R_N(\alpha, \beta) \phi|_{1, \omega} \leq c b_{\alpha, \beta} N^2 \| \phi \|_\omega
$$
with 
$$
b_{\alpha, \beta} = \left( \frac{\Gamma(2 \beta + 1) \Gamma \left( \frac{3}{2} \right)}{\alpha \Gamma(2 \beta + 1 + \frac{3}{2})} \right)^{\frac{1}{2}}.
$$

If in addition $\phi \in \mathcal{P}_N^0$, then 
$$
|R_N^0(\alpha, \beta) \phi|_{1, \omega} \leq c d_{\alpha, \beta} N^2 \| \phi \|_\omega
$$
with 
$$
d_{\alpha, \beta} = \left( \frac{\Gamma(2 \beta + 2) \Gamma \left( \frac{1}{2} \right)}{\alpha \Gamma(2 \beta + 1 + \frac{1}{2})} \right)^{\frac{1}{2}}.
$$

A general form of filtering for Chebyshev approximation is 
$$
R_N \phi(x) = \sum_{i=0}^{N} \sigma_{N,i}^\prime \phi_i(x), \quad \forall \phi \in \mathcal{P}_N
$$
where $0 \leq \sigma_{N,l} \leq 1, \sigma_{N,l} \sim 1$ for $l \ll N$ and $\sigma_{N,l} \ll 1$ for $l \sim N$.

Various methods for recovering the spectral accuracy can also be generalized to Legendre approximation and Chebyshev approximation. For example, Gottlieb and Shu (1995a) developed reconstruction methods for these two approximations. Let $v(x)$ be a function in $L^1(\Lambda)$, which is analytic in a subinterval $[a, b] \subset \Lambda$ and satisfies a condition like (2.35). Its Gegenbauer projection $v_{N,\mu}(x)$ is given by (2.33). Set $\varepsilon = \frac{1}{f}(b - a), \delta = \frac{1}{f}(b + a)$ and
$$
\tilde{w}_{i,\lambda} = \frac{1}{h_{i,\lambda}} \int_{\Lambda} v_{N,\mu}(\varepsilon x + \delta) C_{i,\lambda}(x) \omega_{\lambda}(x) \, dx.
$$

If $\mu \geq 0$ and $\lambda = M = \beta \varepsilon N$ with $\beta < \frac{2}{27}$, then
$$
\max_{|y| \leq 1} \left| v(\varepsilon y + \delta) - \sum_{i=0}^{M} \tilde{w}_{i,\lambda} C_{i,\lambda}(y) \right| \leq b \left( q_1^N + q_2^N \right)
$$
where 
$$
q_1 = \left( \frac{27\beta}{2} \right)^{\beta} < 1, \quad q_2 = \left( \frac{27\varepsilon}{32\rho} \right)^{\beta} < 1,
$$
and $b$ grows at most as $N^{1+2\mu}$. The proof is given in Gottlieb and Shu (1995a). (2.38) gives a reconstructed uniform approximation for Gegenbauer expansion. It covers Legendre approximation and Chebyshev approximation. In Gottlieb and Shu (1996), a sharper result was obtained for Legendre approximation. On the other hand, we can also recover the exponential accuracy from the values of piecewise analytic functions at interpolation points, see Gottlieb and Shu (1995b).
Lecture 3

Stability and Convergence

The theory of stability and convergence plays an important role in numerical solutions of differential equations. Courant, Friedrichs and Lewy (1928) first considered the convergence, and Von Neumann and Goldstine (1947) started the study of stability for linear problems. Kantorovich (1948) developed a general framework for approximations of linear operator equations in Banach space, and Lax and Richtmyer (1956) dealt with linear initial-value problems. For nonlinear problems, Guo (1965), Stetter (1966) and Keller (1975) provided several kinds of stability, which also imply the convergence sometimes.

3.1. Stability and Convergence for Linear Problems

We consider a general framework. Let $B_1$ and $B_2$ be two Banach spaces with the norms $\| \cdot \|_{B_1}$ and $\| \cdot \|_{B_2}$ respectively. For any $v \in B_q$ and $R > 0$, the balls are defined as

$$S_q(v, R) = \{ w \in B_q \mid \| w - v \|_{B_q} < R \}, \quad q = 1, 2.$$  

Let $L$ be a continuous operator from $B_1$ to $B_2$, and $f$ be a given element in $B_2$. Consider the operator equation

$$LU = f. \quad (3.1)$$

In order to solve (3.1) approximately, we first approximate the spaces $B_1, B_2$ and $L, f$. Let $N$ be any positive integer, and $B_{q,N}$ be finite dimensional Banach spaces with the norms $\| \cdot \|_{B_{q,N}}$, $q = 1, 2$. For any $v \in B_{q,N}$ and $R > 0$, the balls are defined by

$$S_{q,N}(v, R) = \{ w \in B_{q,N} \mid \| w - v \|_{B_{q,N}} < R \}, \quad q = 1, 2.$$  

Assume that $\textrm{dim}(B_{1,N}) = \textrm{dim}(B_{2,N})$, and $B_{q,N}$ approximates $B_q$, $q = 1, 2$. Let $p_{q,N}$ be continuous operators from $B_q$ to $B_{q,N}$ such that

$$\lim_{N \to \infty} \| p_{1,N} v \|_{B_{1,N}} = \| v \|_{B_1}, \quad \forall v \in B_1, \quad (3.2)$$
$$\lim_{N \to \infty} \| p_{2,N} v \|_{B_{2,N}} = \| v \|_{B_2}, \quad \forall v \in B_2. \quad (3.3)$$

Let $q_N$ be a continuous operator from $B_{2,N}$ to $B_2$, satisfying

$$p_{2,N} q_N v = v, \quad \forall v \in B_{2,N}. \quad (3.4)$$

We denote by $B_3$ another Banach space with the norm $\| \cdot \|_{B_3}$. $\sigma_N$ is a continuous operator from $B_{1,N}$ to $B_3$, while $\omega$ is a continuous operator from $B_1$ to $B_3$. Suppose
that there exist positive constant $c_0$ and positive integer $N_0$ such that for all $N \geq N_0$,
\[ \|\sigma_N\| = \sup_{v \in B_{1,N}} \frac{\|\sigma_N v\|_{B_2}}{\|v\|_{B_{1,N}}} \leq c_0, \]  
(3.5)
and for any $v \in B_1$,
\[ \lim_{N \to \infty} \|\sigma_N p_{1,N} v - \omega v\|_{B_2} = 0. \]  
(3.6)

In this part, we assume that all operators $L, p_{1,N}, p_{2,N}, q_N, \sigma_N$ and $\omega$ are linear.

Let $L_N$ be an invertible operator from $B_{1,N}$ to $B_{2,N}$. We approximate (3.1) by
\[ L_N u_N = p_{2,N} f. \]  
(3.7)

The approximation error is defined as
\[ R_N (v) = L v - q_N L^N p_{1,N} v, \quad \forall v \in B_1. \]  
(3.8)
Let $D$ be a subset of solutions of (3.1), and $F$ be the set of all $f$ related to $U \in D$. Assume that $F$ is dense in $B_2$. The approximation (3.7) is said to be consistent with (3.1), if for any $v \in D$,
\[ \lim_{N \to \infty} \|R_N (v)\|_{B_2} = 0. \]  
(3.9)

If for all $U$, the solutions of (3.1) and $u_N$, the corresponding solutions of (3.7), there holds
\[ \lim_{N \to \infty} \|\sigma_N u_N - \omega U\|_{B_2} = 0, \]
then we say that (3.7) is convergent.

Usually the data $f$ possesses certain error, denoted by $\hat{f}$. It induces the error of $u_N$, denoted by $\tilde{u}_N$. If there exists a positive constant $c_1$ such that for any $\hat{f}$ and $N \geq N_0$,
\[ \|\sigma_N \tilde{u}_N\|_{B_2} \leq c_1 \|\hat{f}\|_{B_2}, \]  
(3.10)
then we say that (3.7) is stable. Since $L_N$ and $\sigma_N$ are linear, it is equivalent to
\[ \|\sigma_N u_N\|_{B_2} \leq c_1 \|\hat{f}\|_{B_2}. \]  
(3.11)

Since $u_N = L_N^{-1} p_{2,N} f$, it is also equivalent to
\[ \|\sigma_N L_N^{-1} p_{2,N}\|_{B_2} = \sup_{v \in B_{2,N}} \frac{\|\sigma_N L_N^{-1} p_{2,N} v\|_{B_2}}{\|v\|_{B_{2,N}}} \leq c_1. \]  
(3.12)

**Theorem 3.1.** If $L^{-1}$ exists, and (3.7) is consistent with (3.1), then the stability is equivalent to the convergence.

**Theorem 3.2.** If $L^{-1}$ exists, and (3.7) is stable and consistent with (3.1), then
\[ \|\sigma_N u_N - \omega U\|_{B_2} \leq c (\|\sigma_N p_{1,N} U - \omega U\|_{B_2} + \|R_N (U)\|_{B_2}). \]
Theorem 3.1 and Theorem 3.2 can be found in Guo (1988).

If $\omega$ is an isomorphism between $B_1$ and $B_2$, then the above estimate also holds for $\|\omega^{-1} \sigma_N u_N - U\|_{B_1}$.

If $B_3 = B_q = B_{q,N}$, $q = 1,2$, then $p_{1,N}, p_{2,N}, q_N, \sigma_N$ and $\omega$ are the identity operators. If in addition $D = B_1 = B_2 = B_3$, then Theorem 3.1 and Theorem 3.2 turn to be the Basic Convergence Theorem of Kantorovich (1948). Besides, $N'$ could be replaced by any real parameter. It is also noted that in spectral methods, $B_{q,N}$ are certain subspaces of $B_q$, and $p_{q,N}$ are some projections from $B_q$ to $B_{q,N}$, $q = 1,2$, usually. In this case, we can take $B_3 = B_1$ and $\|\cdot\|_{B_3} = \|\cdot\|_{B_1}$, $q = 1,2$. Therefore $q_N, \sigma_N$ and $\omega$ are the identity operators, and so

$$\|u_N - U\|_{B_1} \leq c (\|p_{1,N} U - U\|_{B_1} + \|R_N(U)\|_{B_2}).$$

3.2. Generalized Stability For Nonlinear Problems

In this part, we deal with nonlinear problems. We still consider problem (3.1), but $L$ may be nonlinear. The notations $B_q, B_{q,N}$ and $p_{q,N}$ have the same meanings as before. They also fulfill (3.2) and (3.3), but $p_{1,N}$ and $p_{2,N}$ could be nonlinear. Let $L_N$ and $f_N$ approximate $L$ and $f$ respectively. Consider the approximate problem

$$L_N u_N = f_N.$$  \hfill (3.13)

The approximation error is defined as

$$R_N(v) = p_{2,N} L v - L_N p_{1,N} v + f_N - p_{2,N} f.$$  \hfill (3.14)

If $U$ is a solution of (3.1), then $R_N(U) = f_N - L_N p_{1,N} U$. Let $D$ be the same as before. The approximation (3.13) is consistent with (3.1), if for any $v \in D$ and $f \in F$,

$$\lim_{N \to \infty} \|R_N(v)\|_{B_{1,N}} = 0.$$  \hfill (3.15)

If for all $U$, the solutions of (3.1) and $u_N$, the corresponding solutions of (3.13),

$$\lim_{N \to \infty} \|u_N - p_{1,N} U\|_{B_{1,N}} = 0,$$

then we say that (3.13) is convergent.

How to define the stability is rather a problem. The simplest way is to follow (3.10). It means that there exists a positive constant $c_0$ such that for any $v, w \in B_{1,N}$ and $N \geq N_0$,

$$\|v - w\|_{B_{1,N}} \leq c_0 \|L_N v - L_N w\|_{B_{1,N}}.$$  \hfill (3.16)

Such kind of stability is called linear stability. It is an alternative expression to the stability in the previous section. But nonlinear problems do not possess this property usually. Indeed, this stability is very strong so that it is essentially suitable for linear problems and some specific nonlinear problems, such as Sine-Gordon equation. Firstly, we see that the constant $c_0$ is uniform for all $u_N$, whereas the corresponding constant depends on the exact solution of (3.13) for most nonlinear problems. Next, (3.14) is valid for any $\|L_N v - L_N w\|_{B_{1,N}}$. However, the feature of nonlinear problems is that only when it lies in certain region, the difference between $v$ and $w$ is controlled.
Lecture 3

But \( \|v - w\|_{B_{1,N}} \) may grow rapidly, once \( \|L_N v - L_N w\|_{B_{2,N}} \) exceeds some critical bound. Thus (3.14) can not describe the essence of nonlinear problems properly. Finally, (3.14) holds for all \( v, w \in B_{1,N} \). Clearly this is not possible for nonlinear problems with several solutions. These facts motivated us to look for new definitions of stability. Strang (1965) served the weak stability for linear problems, which is used also in spectral methods, see Gottlieb and Orszag (1977). Stetter (1966) developed the nonlinear stability applied to initial-value problems of nonlinear ordinary differential equations, while Guo (1965, 1982) provided several kinds of generalized stability used for numerical solutions of nonlinear partial differential equations. All of these works are suitable only for problems with unique solutions. Keller (1975) generalized the stability in a different way and proposed the local stability, which has been applied successfully to boundary-value problems of nonlinear ordinary differential equations. In 1985, Guo unified the above work, see Guo (1988). We shall present this general result below.

We first introduce some terminologies. The operator \( L \) is said to be stable in the ball \( S_1(v, R) \), if there are positive constants \( R \) and \( c_R \) such that

\[
\|w - z\|_{B_1} \leq c_R \|Lw - Lz\|_{B_1}, \quad \forall w, z \in S_1(v, R).
\] (3.15)

If \( \bar{U} \) is a solution of (3.1) and there exists a positive constant \( R \) such that \( L \) is stable in \( S_1(U, R) \), then we say that \( U \) is a stable solution. Obviously such a solution is unique in \( S_1(U, R) \). Assume that the data \( f \) in (3.1) possesses the error \( \bar{f} \) which induces the error of solution, denoted by \( \bar{U} \). If the operator \( L \) is linear and stable in the ball \( S_1(U, R) \), then \( \|\bar{U}\|_{B_1} \leq c_R \|\bar{f}\|_{B_1} \). Another kind of solution is called isolated solution. If there exists a linear continuous operator \( L'(v) \) such that

\[
L(v + w) - Lv = L'(v)w + E(v, w)
\]

and

\[
\lim_{\|w\|_{B_1} \to 0} \frac{\|E(v, w)\|_{B_1}}{\|w\|_{B_1}} = 0,
\]

then we say that \( L \) is Fréchet differentiable. \( L'(v) \) is called the Fréchet derivative of \( L \) at the point \( v \). Further, if \( L'(v)w = 0 \) implies \( w = 0 \), then we say that \( L'(v) \) is nonsingular at \( v \). Moreover, if \( U \) is a solution of (3.1) and \( L'(v) \) is nonsingular, then we say that \( U \) is an isolated solution of (3.1). There is a close relation between stable solutions and isolated solutions. Keller (1975) proved that

(i) if \( U \) is a stable solution of (3.1) and \( L'(U) \) exists, then \( U \) is an isolated solution;

(ii) if \( U \) is an isolated solution of (3.1) and \( L'(U) \) is Lipschitz continuous in \( S_1(U, R) \), then \( U \) is a stable solution.

Now, we give the definition of the stability for the approximation (3.13). If there are positive constant \( c_0(v, N) \) and non-negative constant \( c_1(v, N) \) such that for all \( N \geq N_0 \) and \( w \in S_{1,N}(v, R) \), the inequality

\[
\|L_N v - L_N w\|_{B_{2,N}} \leq c_0(v, N)
\] (3.16)

implies

\[
\|v - w\|_{B_{1,N}} \leq c_1(v, N)\|L_N v - L_N w\|_{B_{2,N}},
\] (3.17)
then we say that (3.13) is of generalized weak stability for \( v \) in the ball \( S_{1,N}(v, R) \). Especially, if for all \( N \geq N_0 \) and \( w^{(q)} \in S_{1,N}(v, R) \), the inequalities
\[
\| L_N w^{(q)} - L_N v \|_{B_{2,N}} \leq \varepsilon_0(v, N), \quad q = 1, 2 \tag{3.18}
\]
imply
\[
\| w^{(1)} - w^{(2)} \|_{B_{1,N}} \leq \varepsilon_1(v, N) \| L_N w^{(1)} - L_N w^{(2)} \|_{B_{2,N}}, \tag{3.19}
\]
then we say that (3.13) is of generalized uniform weak stability for \( v \) in the ball \( S_{1,N}(v, R) \).

Theorem 3.3. Let \( L_N \) be a linear operator. The approximation (3.13) is of linear stability, if and only if \( R = \infty \), and \( \varepsilon_0(v, N) \) and \( \varepsilon_1(v, N) \) are independent of \( v \) and \( N \).

We now give a sufficient condition ensuring the generalized uniform weak stability. Let \( L'_N \) denote the Frechet derivative of \( L_N \).

Theorem 3.4. Assume that

(i) \( L_N \) is defined and continuous in \( S_{1,N}(v, R) \);

(ii) for all \( w \in S_{1,N}(v, R) \) such that \( L_N w \in S_{2,N} \), \( (L_N w)^{-1} \) exists and
\[
\| (L_N w)^{-1} \| \leq \varepsilon_1^*(v, N).
\]

Then (3.13) is of generalized uniform weak stability for \( v \) in \( S_{1,N}(v, r(N)) \) where
\[
r(N) = \min \left( \varepsilon_0^*(v, N), \frac{R}{\varepsilon_1^*(v, N)} \right).
\]

We next turn to the existence of solutions of (3.13).

Theorem 3.5. Let \( U \) be a solution of (3.1). Assume that

(i) \( L_N \) is defined and continuous in \( S_{1,N}(p_1 N U, R) \);

(ii) (3.13) is of generalized uniform weak stability for \( p_1 N U \) in \( S_{1,N}(p_1 N U, R) \);

(iii) \( \| R_N(U) \|_{B_{2,N}} < r(N) \) and
\[
r(N) = \min \left( \varepsilon_0(p_1 N U, N), \frac{R}{\varepsilon_1(p_1 N U, N)} \right).
\]

Then (3.13) possesses a unique solution \( u_N \) in the ball \( S_{1,N}(p_1 N U, R) \).

We can evaluate the solution of (3.13) by the Newton procedure
\[
u_N^{(m+1)} = u_N^{(m)} - \left( L'_N \left( u_N^{(m)} \right) \right)^{-1} \left( L_N u_N^{(m)} - f_N \right), \quad m \geq 0,
\]
or the simplified Newton procedure
\[
u_N^{(m+1)} = u_N^{(m)} - \left( L'_N \left( u_N^{(0)} \right) \right)^{-1} \left( L_N u_N^{(m)} - f_N \right), \quad m \geq 0.
\]

It can be verified that both are convergent with the same limit as the solution of (3.13), if the following conditions are fulfilled:
(i) \( L_N \) exists in \( S_{1,N} (u_N^{(0)}, \rho) \), and for all \( \psi \in S_{1,N} (u_N^{(0)}, \rho) \) and certain \( d_N > 0 \),
\[
|| L'_N (\psi^{(1)}) - L'_N (\psi^{(2)}) || \leq d_N || \psi^{(1)} - \psi^{(2)} ||_{B_{1,N}};
\]
(ii) \( g_N = (L'_N (u_N^{(0)}))^{-1} \) exists, and
\[
|| g_N || \leq a_N, \quad || g_N (L_N (u_N^{(0)}) - f_N) || \leq b_N;
\]
(iii) \( \eta_N = a_N b_N d_N < \frac{1}{2} \) and \( \rho \geq \frac{1 - \sqrt{1 - 2 \eta N}}{a_N d_N} \).

We now explain the relation between the stability defined in the above and the stability in actual computation. For instance, let \( \tilde{f}_N \) be the error of \( f_N \) which induces the error of \( u_N \), denoted by \( \tilde{u}_N \). Then we have
\[
L_N (u_N + \tilde{u}_N) = f_N + \tilde{f}_N.
\]
Therefore (3.16) and (3.17) mean that for any \( N \geq N_0 \) and the disturbed solution \( u_N + \tilde{u}_N \) in the ball \( S_{1,N} (u_N, R) \),
\[
|| \tilde{u}_N ||_{B_{1,N}} \leq c_1 (u_N, N) || \tilde{f}_N ||_{B_{2,N}},
\]
provided that
\[
|| \tilde{f}_N ||_{B_{2,N}} \leq c_0 (u_N, N).
\]
If \( L^{-1} \) exists, then (3.13) has a unique solution and \( R = \infty \). If, in addition, \( c_0 (u_N, N) \)
is arbitrary and \( c_1 (u_N, N) = c_1 (u_N) \), then we get the best stability for usual nonlinear problems. It means that there is no restriction on the data error. In this case, this stability is nearly the same as the linear stability. But they are still different, since \( c_1 (u_N) \) depends on \( u_N \). If \( c_0 (u_N, N) = c_0 (u_N) \) and \( c_1 (u_N, N) = c_1 (u_N) \), then the error of solution is bounded by the data error provided that the data error does not exceed a critical value. Such stability was also investigated by Stetter (1966). Further, if \( c_0 (u_N, N) = c_0 (u_N) N^{-s'} \) and \( c_1 (u_N, N) = c_1 (u_N) \), then it turns out to be the generalized stability of Guo (1965, 1982). It is also called g-stability. The infimum of such values \( s' \) is called the index of g-stability. On the other hand, if \( c_0 (u_N, N) \)
is arbitrary and \( c_1 (u_N, N) = cN^q \), then it is a generalization of the weak stability of Strang (1965). Finally if \( R < \infty \), \( c_1 (u_N, N) = c_1 (u_N) \) and \( c_0 (u_N, N) \) is arbitrary, then it is equivalent to the local stability by Keller (1975). It means that in the ball \( S_{1,N} (u_N, R) \), the approximation (3.13) is locally stable.

Under some conditions, the above stability implies the convergence.

**Theorem 3.6.** Let \( U \) and \( u_N \in S_{1,N}(p_1,NU,R) \) be the solutions of (3.1) and (3.13) respectively. Suppose that

(i) (3.13) is of generalized weak stability for \( p_1,NU \) in \( S_{1,N}(p_1,NU,R) \);
(ii) \( || R_N(U) ||_{B_{2,N}} \leq c_0(p_1,NU,N) \).
Then
\[ \|u_N - p_{1,N} U\|_{B_{1,N}} \leq c_1 (p_{1,N} U, N) \| R_N(U) \|_{B_{2,N}}. \]

We know from Theorem 3.6 that (3.13) is convergent, if \( c_1 (p_{1,N} U, N) \| R_N(U) \|_{B_{2,N}} \to 0 \) as \( N \to \infty \). In particular, if (3.13) is of the generalized stability with the index \( s \) and \( \| R_N(U) \|_{B_{2,N}} \leq c_2 (U) N^{-q}, q > \max(s,0) \), then
\[ \| u_N - p_{1,N} U\|_{B_{1,N}} \leq c_2 (U) N^{-q}. \]

### 3.3. Initial-Value Problems

We now focus on initial-value problems. Let \( B_1 \) and \( B_2 \) be two Banach spaces with the norms \( \| \cdot \|_{B_1} \) and \( \| \cdot \|_{B_2} \), \( B_1 \subseteq B_2 \), \( Q \) and \( L \) are two differential operators with respect to the spatial variables, mapping \( v \in B_1 \) into \( B_2 \). For simplicity, assume that \( Q \) is a linear invertible operator. Denote by \( B_3 \) the third Banach space with the norm \( \| \cdot \|_{B_3} \), whose element is a mapping from the interval \( (0,T) \) to \( B_2, T > 0 \). Let \( f(t) \) be a given element in \( B_3 \) and \( U_0 \) be a given element in \( B_1 \). \( U_0 \) describes the initial state. Consider the operator equation
\[
\begin{align*}
\begin{cases}
\partial_t (QU(t)) = LU(t) + f(t), & 0 < t \leq T, \\
U(0) = U_0,
\end{cases}
\end{align*}
\]  
(3.20)

We define a genuine solution of (3.20) as a one-parameter family \( U(t) \) such that \( U(t) \) is in the domain of \( Q \) and \( L \) for each \( t \in [0,T] \), and
\[ \left\| \frac{1}{s} (QU(t + s) - QU(t)) - LU(t) - f(t) \right\|_{B_2} \to 0, \text{ as } s \to 0. \]

Let \( D_1 \) and \( D_3 \) be such subsets of \( B_1 \) and \( B_3 \) that for any \( U_0 \in D_1 \) and \( f \in D_3 \), (3.20) has a unique solution. Suppose that \( D_1 \) and \( D_3 \) are dense in \( B_1 \) and \( B_3 \) respectively. The corresponding subset of solutions is denoted by \( D_2 \).

Now let \( N \) be any positive integer and \( B_{q,N} \) be finite dimensional Banach spaces with the norms \( \| \cdot \|_{B_{q,N}} \), approximating the spaces \( B_q, q = 1,2,3 \). In addition, \( \dim(B_{1,N}) = \dim(B_{2,N}) \). Denote by \( p_{q,N} \) the continuous operators from \( B_q \) to \( B_{q,N} \), satisfying the following conditions:

(i) for all \( v \in B_1, p_{1,N} v = p_{2,N} v; \)

(ii) for all \( v \in B_q, \| p_{q,N} v \|_{B_{q,N}} \to \| v \|_{B_q} \) as \( N \to \infty ,q = 1,2,3 \).

Next, let \( \tau = \tau(N) > 0 \) be the mesh spacing of the variable \( t \), and \( D_t v \) be the forward difference quotient with respect to \( t \), defined in Lecture 1.

We now approximate \( QU(t), LU(t), f(t) \) and \( U_0 \) by \( Q_N u_N(t), L_N(u_N(t), u_N(t + \tau)), f_N(t) \) and \( u_{N,0} \) respectively, and then consider the approximate problem
\[
\begin{align*}
\begin{cases}
D_t (Q_N u_N(t)) = L_N (u_N(t), u_N(t + \tau)) + f_N(t), \\
u_N(0) = u_{N,0}.
\end{cases}
\end{align*}
\]  
(3.21)

Suppose that for each \( t, u_N(t) \) is determined uniquely by (3.21). The simplest case is that \( Q_N^{-1} \) exists and \( L_N (u_N(t), u_N(t + \tau)) \) is independent of \( u_N(t + \tau) \). The
approximation error for $t > 0$ is defined as
\[
R_N(v(t)) = -p_{2,N}q_N v(t) + D_N l_N p_{1,N} v(t) + p_{2,N} L v(t) - L_N (p_{1,N} v(t), p_{1,N} v(t + \tau)) + p_{2,N} f(t) - f_N(t).
\]

If $U(t)$ is the solution of (3.20) and $p_{2,N}$ is linear, then $R_N(U(t))$ is reduced to
\[
R_N(U(t)) = D_N Q N p_{1,N} U(t) - L_N (p_{1,N} U(t), p_{1,N} U(t + \tau)) - f_N(t).
\]
The approximation (3.21) is said to be consistent with (3.20), if for any $U_0 \in D_1, U \in D_2, f \in D_3$ and $0 \leq t \leq T$,
\[
\|p_{1,N} U_0 - u_{N,0}\|_{B_{1,N}} \rightarrow 0, \quad \|R_N(U(t))\|_{B_{2,N}} \rightarrow 0, \quad \text{as } N \rightarrow \infty.
\]
If for all $U(t)$, the solutions of (3.20) and $u_N(t)$, the corresponding solutions of (3.21),
\[
\|p_{1,N} U(t) - u_N(t)\|_{B_{1,N}} \rightarrow 0, \quad 0 \leq t \leq T, \quad \text{as } N \rightarrow \infty,
\]
then we say that the approximation (3.21) is convergent.

We now turn to the stability. We first consider the linear stability. It means that there is a positive constant $c_0(T)$ such that for all $N \geq N_0$ and $0 \leq t \leq T$,
\[
\|u^{(1)}_N(t) - u^{(2)}_N(t)\|_{B_{1,N}} \leq c_0(T) \left(\|u^{(1)}_{N,0} - u^{(2)}_{N,0}\|_{B_{1,N}} + \|f^{(1)}_N - f^{(2)}_N\|_{B_{2,N}}\right) \quad (3.22)
\]
where $u^{(q)}_N$ are the solutions of (3.21), corresponding to $u^{(q)}_{N,0}$ and $f^{(q)}_N$, $q = 1, 2$.

**Theorem 3.7.** If (3.20) is a well-posed linear problem and (3.21) is consistent with (3.20), then the linear stability is equivalent to the convergence. Moreover for all $0 \leq t \leq T$,
\[
\|u_N(t) - p_{1,N} U(t)\|_{B_{1,N}} \leq c_0(T) \left(\|u_{N,0} - p_{1,N} U_0\|_{B_{1,N}} + \|R_N(U)\|_{B_{2,N}}\right).
\]

We now consider nonlinear problems. If there are positive constant $c_0(u^{(1)}_N, N, T)$ and non-negative constant $c_1(u^{(1)}_N, N, T)$ such that for all $N \geq N_0$, the inequality
\[
\|u^{(1)}_{N,0} - u^{(2)}_{N,0}\|_{B_{1,N}} + \|f^{(1)}_N - f^{(2)}_N\|_{B_{2,N}} \leq c_0 \left(u^{(1)}_N, N, T\right)
\]
implies
\[
\|u^{(1)}_N(t) - u^{(2)}_N(t)\|_{B_{1,N}} \leq c_1 \left(u^{(1)}_N, N, T\right) \left(\|u^{(1)}_{N,0} - u^{(2)}_{N,0}\|_{B_{1,N}} + \|f^{(1)}_N - f^{(2)}_N\|_{B_{2,N}}\right), \quad 0 \leq t \leq T,
\]
u^{(q)}_N(t) being the solutions of (3.21) related to $u^{(q)}_{N,0}$ and $f^{(q)}_N$, $q = 1, 2$, then we say that the approximation (3.21) is of generalized weak stability at $u^{(1)}_N$.

There is a close relation between the generalized weak stability and the convergence, which extends the result of Lax Theorem, see Guo (1994).
**Theorem 3.8.** Let $U(t)$ and $u_N(t)$ be the solutions of (3.20) and (3.21) respectively. If (3.21) is of generalized weak stability at $p_{1,N}U(t)$ and

$$
\|p_{1,N}U_0 - u_{N,0}\|_{B_{1,N}} + \|R_N(U)\|_{B_{3,N}} \leq c_0(p_{1,N}U, N, T),
$$

then for all $0 \leq t \leq T$,

$$
\|p_{1,N}U(t) - u_N(t)\|_{B_{1,N}} \leq c_1(p_{1,N}U, N, T) \left( \|p_{1,N}U_0 - u_{N,0}\|_{B_{1,N}} + \|R_N(U)\|_{B_{3,N}} \right).
$$

The simplest case of generalized weak stability is also called $g$-stability. In this case,

$$
c_0(u_N, N, T) = c_0(u_N, T) N^{-s'}, \quad c_1(u_N, N, T) = c_1(u_N, T).
$$

The infimum of such values $s'$ is also called the index of $g$-stability, denoted by $s$. Clearly, if $u_{N,0} = p_{1,N}U_0$, $\|p_{1,N}U_0 - U_0\|_{B_{1,N}} = \mathcal{O}(N^{-q})$, $\|R_N(U)\|_{B_{3,N}} = \mathcal{O}(N^{-q})$, $q > 0$ and $s < q$, then (3.21) is convergent with the accuracy $\mathcal{O}(N^{-q})$. 

Lecture 4

Spectral Methods and Pseudospectral Methods

In this lecture, we apply various orthogonal approximations in Sobolev spaces to numerical solutions of partial differential equations. Particular attention is paid to nonlinear problems arising in fluid dynamics, quantum mechanics and some other topics. Some filterings are adopted in pseudospectral methods for the improvement of stability. Several spectral penalty methods are introduced for initial-boundary value problems with inhomogeneous data on the boundary. We also describe vanishing viscosity methods for simulating shock waves. Finally, we discuss the spectral approximations to isolated solutions of nonlinear problems.

4.1. Fourier Spectral Methods and Fourier Pseudospectral Methods

We take Benjamin-Bona-Mahony equation (BBM equation) as an example to show how to construct Fourier spectral schemes and Fourier pseudospectral schemes. The BBM equation describes the movement of regularized long waves. It is of the form

\[
\begin{align*}
\partial_t U(x,t) + d \partial_x U(x,t) + U(x,t) \partial_x U(x,t) \\
-\delta \partial_t \partial_x^2 U(x,t) = f(x,t), \\
U(x,0) = U_0(x),
\end{align*}
\]

where \(U_0\) and \(f\) are given functions, and \(d \geq 0\), \(\delta > 0\). Moreover \(U_0\) and \(f\) possess the period \(2\pi\) for the variable \(x\), and so does the solution \(U\). Let \(\Lambda = (0,2\pi)\) and denote the space \(C(0,T; H^r_p(\Lambda))\) by \(C(0,T; H^r_p)\) for simplicity. Also \(\|\cdot\|_r = \|\cdot\|_{H^r(\Lambda)}\) etc. It can be verified that if \(U_0 \in H^r_p(\Lambda)\) and \(f \in L^2(0,T; L^2_p)\), then (4.1) has a unique solution \(U \in L^\infty(0,T; H^r_p)\).

It is commonly admitted that a reasonable numerical algorithm should preserve the features of the genuine solution of the original problem as much as possible. In fact, the solution of (4.1) possesses some conservations, such as

\[
\int_\Lambda U(x,t) \, dx = \int_\Lambda U_0(x,t) \, dx + \int_0^t \int_\Lambda f(x,s) \, dx \, ds,
\]

\[
\|U(t)\|_2^2 + \delta \|U(t)\|_2^2 = \|U_0\|_2^2 + \delta \|U_0\|_2^2 + 2 \int_0^t \langle f(s), U(s) \rangle \, ds.
\]

We are going to construct the Fourier spectral scheme. Let \(N\) be any positive integer and \(V_N\) be the subspace of all real-valued functions of \(\tilde{V}_N\) by (2.1). Denote by \(P_N\) the \(L^2\)-orthogonal projection from \(L^2_p(\Lambda)\) onto \(V_N\). For convenience of analysis,
let
\[ J(v(x), w(x)) = \frac{1}{3} w(x) \partial_x v(x) + \frac{1}{3} \partial_x (w(x)v(x)). \]

It is shown that
\[ (J(v, w), z) + (J(z, w), v) = 0, \quad (4.4) \]
\[ (J(v, w), w) = \frac{1}{2} (\partial_x v, w^2). \quad (4.5) \]

For discretization in time \( t \), let \( \tau \) be the mesh size and use the notations in Lecture 1, such as \( R_\tau(T), \tilde{R}_\tau(T) \) and \( D_x v(x, t), \) etc.

Let \( u_N \) be the approximation to \( U \), and \( \sigma_i \) be parameters, \( 0 \leq \sigma_i \leq 1, i = 1, 2, 3 \).

A Fourier spectral scheme for (4.1) is to find \( u_N(x, t) \in V_N \) for all \( t \in \tilde{R}_\tau(T) \) such that
\[
\begin{aligned}
D_x u_N(x, t) + \sigma_1 d \partial_x u_N(x, t + \tau) + (1 - \sigma_1) d \partial_x u_N(x, t) \\
+ \sigma_2 P_N J(u_N(x, t + \tau), u_N(x, t)) + (1 - \sigma_2) P_N J(u_N(x, t), u_N(x, t)) \\
- \delta D_x \partial_x^2 u_N(x, t) = F_N(x, t), \quad x \in \mathbb{R}^1, \ t \in R_\tau(T), \\
u_N(x, 0) = u_{N_0}(x), \quad x \in \mathbb{R}^1;
\end{aligned}
\quad (4.6)
\]

where \( u_{N_0}(x) = P_N U_0(x) \) and
\[ F_N(x, t) = \sigma_1 P_N f(x, t + \tau) + (1 - \sigma_1) P_N f(x, t). \]

If \( \sigma_1 = \sigma_2 = \sigma_3 = 0 \), then (4.6) is the explicit spectral scheme in Guo and Manoranjan (1985). If \( \sigma_1 \neq 0 \) and \( \sigma_2 = 0 \), it becomes an implicit scheme. But by the orthogonality of Fourier system, we still can evaluate the coefficients of \( u_N(x, t) \) explicitly at each time \( t \in R_\tau(T) \). This shows the merit of spectral methods. In (4.6), the nonlinear term is approximated by partially implicit approach and so we only have to solve a linear system for the Fourier coefficients step by step, even if \( \sigma_1 \sigma_2 \neq 0 \). It can be proved that the solution is unique.

We check the conservations. If \( \sigma_2 = 0 \), then
\[
\int_{\Lambda} u_N(x, t) \, dx = \int_{\Lambda} u_{N_0}(x, t) \, dx + \tau \sum_{s \in \tilde{R}_\tau(1-\tau)} \int_{\Lambda} F_N(x, s) \, dx,
\]

If \( \sigma_1 = \sigma_2 = \frac{1}{2} \), then
\[ ||u_N(t)||^2 + \delta ||u_N(t)||^2 = ||u_{N_0}||^2 + \delta ||u_{N_0}||^2 + \tau \sum_{s \in \tilde{R}_\tau(1-\tau)} (F_N(s), u_N(s + \tau) + u_N(s)). \]

The above two equations are reasonable analogues of the conservations (4.2) and (4.3).

We now turn to analyze the errors. Suppose that \( u_{N_0} \) and \( F_N \) have the errors \( \tilde{u}_{N_0} \) and \( \tilde{F}_N \) respectively. Then we get a disturbed solution corresponding to \( u_{N_0} + \tilde{u}_{N_0} \) and \( F_N + \tilde{F}_N \), denoted by \( u_N + \tilde{u}_N \). For simplicity, we denote the errors \( \tilde{u}_N, \tilde{u}_{N_0} \) and \( \tilde{F}_N \) by \( \tilde{u}, \tilde{u}_0 \) and \( \tilde{F} \). They satisfy the error equation as follows
\[
\begin{aligned}
D_x \tilde{u}(x, t) + d \partial_x \tilde{u}(x, t) + \sigma_1 d \tau D_x \partial_x \tilde{u}(x, t) + P_N J(u_N(x, t) \\
+ \sigma_2 \tau D_x u_N(x, t), \tilde{u}(x, t)) + P_N J(\tilde{u}(x, t), \sigma_2 \tau D_x \tilde{u}(x, t), u_N(x, t) + \tilde{u}(x, t)) \\
- \delta D_x \partial_x^2 \tilde{u}(x, t) = \tilde{F}(x, t), \quad x \in \mathbb{R}^1, \ t \in \tilde{R}_\tau(T).
\end{aligned}
\quad (4.7)
\]
For describing the average error of numerical solution, we let $q_0 \geq 0$ and define
\begin{equation*}
E(v, t) = \|v(t)\|^2 + \delta |v(t)|^2 + q_0 \tau^2 \sum_{s \in R(t - \tau)} (\|D_r v(s)\|^2 + \delta |D_r v(s)|^2).
\end{equation*}
Also for the average error of data, set
\begin{equation*}
\rho(v, w, t) = \|w\|^2 + \delta |v|^2 + \tau \sum_{s \in R(t - \tau)} |w(s)|^2.
\end{equation*}
By using (4.4) and (4.5), we can prove the following result.

**Theorem 4.1.** There exist positive constants $b_1$ and $b_2$ depending only on $d, \delta$ and $\|u_0\|_{C([0,T;H^{-1})}, \tau > \frac{3}{2}$, such that if for certain $t_1 \in \overline{R}(T), \rho(\bar{u}_0, \bar{F}, t_1) \leq \frac{b_1}{\tau N}$, then for all $t \in \overline{R}(t_1)$,
\begin{equation*}
E(\bar{u}, t) \leq c \rho(\bar{u}_0, \bar{F}, t)e^{b_2 t}.
\end{equation*}
If in addition $\sigma_2 > \frac{1}{2}$, then the above estimate holds for all $t \in \overline{R}(T)$ and any $\rho(\bar{u}_0, \bar{F}, t)$.

Theorem 4.1 shows that scheme (4.6) is of the generalized stability with the index $s \leq 1 - q$, provided that $\tau = O(N^{-q})$. If $q = 1$, then $s = 0$. In this case, the computation is stable when the average error of data does not exceed a critical value $cb_1$. If $\sigma_2 > \frac{1}{2}$, then there is no restriction on $\rho(\bar{u}_0, \bar{F}, t)$. It means that scheme (4.6) possesses the index of generalized stability $s = -\infty$. We know from the above analysis that the suitable choice of parameter in the approximation of nonlinear term can improve the stability essentially.

We next deal with the convergence of (4.6). Let $\bar{U}_N = P_N \bar{U}$ and $\sigma_3 = 0$ for simplicity. By (4.1),
\begin{equation*}
\begin{cases}
D_{\tau} U_N (x, t) + d \partial_x U_N (x, t) + \alpha_1 d_{\tau} D_{\tau} \partial_x U_N (x, t) \\
+ P_N J (U_N (x, t), U_N (x, t), U_N (x, t), U_N (x, t)) = F_N (x, t) + \sum_{j=1}^5 G_j (x, t), \quad x \in \mathbb{R}, t \in \overline{R}(T),
\end{cases}
\end{equation*}
where
\begin{align*}
G_1 (x, t) &= D_{\tau} U_N (x, t) - \partial_x U_N (x, t), \\
G_2 (x, t) &= \alpha_1 d_{\tau} D_{\tau} \partial_x U_N (x, t), \\
G_3 (x, t) &= P_N J (U_N (x, t), U_N (x, t)) - P_N J (\bar{U} (x, t), \bar{U} (x, t)), \\
G_4 (x, t) &= \alpha_2 \tau P_N J (U_N (x, t), U_N (x, t)), \\
G_5 (x, t) &= \partial_x \partial^2_x U_N (x, t) - D_{\tau} \partial_{\tau}^2 U_N (x, t).
\end{align*}
Further, let $ar{U}_N (x, t) = u_N (x, t) - U_N (x, t)$, also denoted by $\bar{U} (x, t)$ for simplicity. By (4.6), it follows that
\begin{equation*}
\begin{aligned}
D_{\tau} \bar{U} (x, t) + d \partial_x \bar{U} (x, t) + \alpha_1 d_{\tau} D_{\tau} \partial_x \bar{U} (x, t) + P_N J (U_N (x, t) \\
+ \alpha_2 \tau D_{\tau} \bar{U} (x, t), \bar{U} (x, t)) + P_N J (\bar{U} (x, t) + \alpha_2 \tau D_{\tau} \bar{U} (x, t), \bar{U} (x, t)) \\
+ \bar{U} (x, t) - D_{\tau} \partial^2_{\tau} \bar{U} (x, t) = - \sum_{j=1}^5 G_j (x, t).
\end{aligned}
\end{equation*}
In addition, \( \hat{U}(x, 0) = 0 \). By comparing this error equation to (4.7), we find that it suffices to estimate the terms \( G_j(x, t), 1 \leq j \leq 5 \). Finally we get from Theorem 4.1 the following result.

**Theorem 4.2.** Let \( \sigma_2 = 0 \) in (4.6). If \( U \in C(0, T; H^p_0) \cap H^2(0, T; H^1), r > \frac{3}{2} \) and \( \tau \leq b_3N^{-\frac{1}{r}} \), then for all \( t \in \bar{R}_\tau(T) \),

\[
E(U_N - u_N, t) \leq b_4 e^{b_5 \tau} \left( \tau^2 + N^{2-2r} \right)
\]

where \( b_3, b_4 \) and \( b_5 \) are positive constants depending on \( d, \delta, ||U||_{C(0, T; H^p_0)} \) and \( ||U||_{H^2(0, T; H^1)} \).

If \( \sigma_2 > \frac{1}{2} \), then the above estimate is valid for any \( \tau > 0 \).

As indicated before, it is not easy to deal with nonlinear terms in spectral methods usually. So we prefer Fourier pseudospectral methods in actual computations. Let \( x^{(j)}, \lambda_N \) and \( I_N \) be the Fourier interpolation points, the discrete interval and the Fourier interpolant in (2.3) and (2.4). The discrete inner product \( \langle \cdot, \cdot \rangle_N \) and the discrete norm \( ||\cdot||_N \) are given by (2.5).

We can approximate (4.1) directly and derive a scheme similar to (4.6), in which the projection \( P_N \) is replaced by \( I_N \). But such a scheme cannot provide good numerical results. This is caused by two things. Firstly, the conservations (4.2) and (4.3) are simulated in spectral scheme (4.6) automatically. However this is not so in pseudospectral methods, since interpolation is used. In order to preserve the global properties of genuine solution of (4.1), we have to deal with the nonlinear term carefully. For this purpose, we define the operator

\[
J_N(v, w)(x) = \frac{1}{3} I_N(w \partial_x v)(x) + \frac{1}{3} \partial_x I_N(wv)(x).
\]

Let \( \phi, \psi \in V_N \). Then

\[
(J_N(\phi, w), \psi) + (J_N(\psi, w), \phi) = 0. \tag{4.9}
\]

Therefore \( J_N(\phi, w) \) preserves the property of \( J(\phi, w) \). This technique is called skew symmetric decomposition of nonlinear convective term. It enables the numerical solution to keep the conservations. On the other hand, the nonlinear computation may destroy the stability. Mainly, this trouble is due to the higher frequency terms. To remedy this, we can use the filtering \( R_N(\alpha, \beta) \) in (2.28), denoted by \( R_N \) for simplicity.

A Fourier pseudospectral scheme for (4.1) is to find \( u_N(x, t) \in V_N \) for all \( t \in \bar{R}_\tau(T) \) such that

\[
\begin{cases}
D_t u_N(x, t) + dR_N \partial_x u_N(x, t) + \sigma_1 d\tau R_N D_x \partial_x u_N(x, t) + R_N J_N(R_N u_N(x, t) \\
+ \sigma_2 \tau R_N D_x u_N(x, t), u_N(x, t)) - \delta D_x \partial_x^2 u_N(x, t) \\
= F_N(x, t), & x \in \mathbb{R}^1, t \in \bar{R}_\tau(T),
\end{cases}
\]

where \( u_{N,0}(x) = I_N U_0(x) \) and

\[
F_N(x, t) = R_N I_N(f(x, t) + \sigma_3 \tau D_x f(x, t)).
\]

Clearly for any \( \phi \in V_N \),

\[
(R_N \partial_x \phi, \phi) = \left( \partial_x R_N^+ \phi, R_N^+ \phi \right) = 0. \tag{4.11}
\]
By (4.9), for any $\phi, \psi \in V_N$,
\[
(R_N J_N (R_N \phi, \psi) + (R_N J_N (R_N \psi, \phi) = 0. \tag{4.12}
\]
If $\sigma_2 = 0$, then
\[
\int \Lambda \ u_N(x, t) \ dx = \int \Lambda \ u_{N,0}(x) \ dx + \tau \sum_{s \in \bar{R}_{\tau}(t, -\tau)} \int \Lambda \ F_N(x, s) \ dx.
\]
If $\sigma_1 = \sigma_2 = \frac{1}{2}$, then it follows from (4.11) and (4.12) that
\[
\|u_N(t)\|^2 + \delta |u_N(t)\|^2 = \|u_{N,0}\|^2 + \delta |u_{N,0}\|^2 + \tau \sum_{s \in \bar{R}_{\tau}(t, -\tau)} (F_N(s), u_N(s + \tau) + u_N(s)).
\]
The above two equalities are reasonable analogies of (4.2) and (4.3).

We next analyze the errors. Let $\hat{u}_0$ and $\hat{F}$ be the errors of $u_{N,0}$ and $F_N$, respectively, which induce the error of $u_N$, denoted by $\hat{u}$. Then
\[
D_{\tau} \hat{u}(x, t) + dR_N \partial_x \hat{u}(x, t) + \sigma_1 d\tau R_N D_{\tau} \partial_x \hat{u}(x, t) + R_N J_N (R_N u_N(x, t) + u_2 R_N D_{\tau} \partial_x \hat{u}(x, t) + R_N J_N (R_N \hat{u}(x, t) + \sigma_2 R_N D_{\tau} \hat{u}(x, t), u_N(x, t) + \hat{u}(x, t)) - \delta D_{\tau} \partial_x^2 \hat{u}(x, t) = \hat{F}(x, t), \quad x \in \mathbb{R}^d, t \in \bar{R}_{\tau}(T).
\]

By a careful analysis, we obtain the following conclusion.

**Theorem 4.3.** There exist positive constants $b_0$ and $b_7$ depending only on $d, \delta, \alpha, \beta$ and $\|u_N\|_{C(0, T; H^r)}, r > \frac{3}{2}$, such that if for certain $t_1 \in R_\tau(T), \rho(\hat{u}_0, \hat{F}, t_1) \leq \frac{b_0}{\tau N}$, then for all $t \in \bar{R}_{\tau}(t_1)$,
\[
E(\hat{u}, t) \leq c\rho(\hat{u}_0, \hat{F}, t) e^{b r t}.
\]

If in addition $\sigma_2 > \frac{1}{2}$, then the above estimate holds for all $t \in \bar{R}_{\tau}(T)$. and any \( \rho \left( \hat{u}_0, \hat{F}, t \right) \).

We now turn to the convergence of (4.10). Let $\bar{U}_N = P_N U, \bar{U} = u_N - U_N$ and $\sigma_3 = 0$ in (4.10). We obtain the error equation as follows
\[
D_{\tau} \bar{U}(x, t) + dR_N \partial_x \bar{U}(x, t) + \sigma_1 d\tau R_N D_{\tau} \partial_x \bar{U}(x, t) + R_N J_N (R_N \bar{U}(x, t) + \sigma_2 R_N D_{\tau} \bar{U}(x, t), U_N(x, t) + \bar{U}(x, t)) - \delta D_{\tau} \partial_x^2 \bar{U}(x, t) = - \sum_{j=1}^{6} G_j(x, t),
\]
where $G_1(t)$ and $G_5(t)$ are the same as in (4.8), and
\[
G_2(t) = \sigma_1 d\tau R_N D_{\tau} \partial_x U_N(x, t),
\]
\[
G_3 = R_N J_N (R_N \bar{U}(x, t), U_N(x, t)) - P_N J(U(x, t), U(x, t)),
\]
\[
G_4(t) = \sigma_2 R_N J_N (R_N D_{\tau} \bar{U}(x, t), U_N(x, t)),
\]
\[
G_6 = P_N f(x, t) - R_N I_N f(x, t).
\]

By Theorem 2.4, Theorem 2.26, imbedding theory and (4.12), we obtain the following result.
Theorem 4.4. Let \( \sigma_3 = 0 \) in (4.10). If \( U \in C(0,T;H^p_p) \cap H^2(0,T;H^1), U_0 \in H^p_p(\Lambda), f \in C(0,T;H^p_p), \alpha \geq r > \frac{3}{2} \) and \( \tau \leq b_N^{-\frac{1}{2}} \), then for all \( t \in \tilde{R}_\tau(T) \),

\[
E(U_N - u_N, t) \leq b_N e^{b_N t} (\tau^2 + N^{2-2r})
\]

where \( b_0 - b_N \) are positive constants depending on \( d, \delta, \alpha, \beta, ||U||_{C(0,T;H^p_p)}, ||U||_{H^2(0,T;H^1)}, ||U_0||_r \) and \( ||f||_{C(0,T;H^p_p)} \). If \( \sigma_2 > \frac{1}{2} \), then the conclusion is valid for any \( \tau > 0 \).

Scheme (4.10) with \( \sigma_1 = \sigma_2 = \sigma_3 = 0 \) was investigated in Guo and Cao (1988). Since the nonlinear term \( U \partial_x U \) is approximated by a skew symmetric operator \( R_N J_N (R_N u_N, u_N) \), the effects of the leading nonlinear error terms are cancelled, i.e.,

\[
\begin{align*}
(R_N J_N (R_N \hat{u}(t), \hat{u}(t)), \hat{u}(t)) &= 0, \\
\left( R_N J_N (R_N \hat{U}(t), \hat{U}(t)), \hat{U}(t) \right) &= 0.
\end{align*}
\]

Otherwise, we shall require that \( \rho(\hat{u}_0, \hat{f}_0, t_1) \leq \frac{b_0}{N} \) in Theorem 4.3. It indicates that the suitable approximation to the nonlinear term, not only enables the scheme to keep the conservations, but also strengthens the stability and raises the accuracy. It also shows that there is a close relation between the conservations and the stability. This idea was first proposed for finite difference methods by Guo (1965).

4.2. Legendre Spectral Methods And Legendre Pseudospectral Methods

For non-periodic problems, it is natural to use Legendre spectral methods or Legendre pseudospectral methods. Because of the appearance of the Fast Legendre Transformation, these two methods have been paid more and more attention recently. We take the nonlinear Klein-Gordon equation (NLKG equation) as an example to describe these two methods. The NLKG equation plays an important role in quantum mechanics. Let \( \Lambda = (-1,1) \). The initial-boundary value problem of NLKG equation is as follows

\[
\begin{align*}
\partial_t^2 U(x,t) + U(x,t) + U^3(x,t) - \partial_x^2 U(x,t) &= f(x,t), & x \in \Lambda, 0 < t \leq T, \\
U(-1,t) &= U(1,t) = 0, & 0 < t \leq T, \\
\partial_t U(x,0) &= U_1(x), & x \in \Lambda, \\
U(x,0) &= U_0(x), & x \in \Lambda.
\end{align*}
\]

(4.13)

It is shown that if \( U_0 \in H^3_0(\Lambda) \cap L^4(\Lambda), U_1 \in L^2(\Lambda) \) and \( f \in L^2(0,T;L^2) \), then (4.13) has a unique solution \( U \in L^\infty \left(0,T;H^3_0 \cap L^4(\Lambda) \right) \cap W^1,\infty(0,T;L^2) \). Let

\[
E^*(v,0) = ||\partial_t v(t)||^2 + ||v(t)||^2 + \frac{1}{2} ||v(t)||_{L^2}^4.
\]

The solution \( U \) satisfies the conservation

\[
E^*(U, t) = E^*(U, 0) + 2 \int_0^t (f(s), \partial_x U(s)) \, ds.
\]

(4.14)
Lecture 4

For Legendre spectral approximation, let \( N \) be any positive integer and \( \mathbb{P}_N, \mathbb{P}_0 \) be as before. Denote by \( P_N^0 \) the \( L^2 \)-orthogonal projection from \( L^2(\Lambda) \) onto \( \mathbb{P}_N^0 \). The \( H_0^1 \)-orthogonal projection \( P_N^{1,0} \) is given by (2.17). Also let \( \tau \) be the mesh size in \( t \) and use the notations \( R_\tau(T), \bar{R}_\tau(T), D_\tau v(x,t), \bar{D}_\tau v(x,t) \) and \( \bar{D}_\tau v(x,t) \) in Lecture 1.

We try to construct a scheme preserving the conservation (4.14). To do this, define

\[
G(v(x,t)) = \frac{1}{4} \sum_{j=0}^{3} v^j(x,t + \tau)v^{3-j}(x,t - \tau).
\]

Clearly

\[
\left( G(v(t)), \bar{D}_\tau v(t) \right) = \frac{1}{4} \bar{D}_\tau \|v(t)\|_{L^4}^4. \tag{4.15}
\]

Let \( u_N \) be the approximation to \( U \). We approximate the nonlinear term \( U^3 \) by \( P_N G(u_N) \) instead of \( P_N U^3 \). A Legendre spectral scheme for (4.13) is to find \( u_N(x,t) \in \mathbb{P}_N^0 \) for all \( t \in \bar{R}_\tau(T) \) such that

\[
\left\{ \begin{array}{l}
\bar{D}_\tau u_N(t) + \bar{u}_N(t) + G(u_N(t)) = \phi(t), \quad \forall \phi \in \mathbb{P}_N^0, t \in R_\tau(T), \\
D_\tau u_N(0) = u_{N,1}, \\
u_N(0) = u_{N,0}
\end{array} \right. \tag{4.16}
\]

where \( u_{N,0} = P_N^0 U_0 \) and

\[
u_{N,1} = P_N U_1 + \frac{1}{2} \tau P_N \left( \partial_\tau^2 U_0 - U_0 - U^3_0 + f(0) \right) .
\]

Let

\[
E_\tau^*(v,t) = \left\| D_\tau v(t) \right\|_2^2 + \frac{1}{2} \left\| v(t) \right\|_{H_0^1}^2 + \frac{1}{2} \left\| v(t - \tau) \right\|_{H_0^1}^2 + \frac{1}{4} \left\| v(t) \right\|_{L^4}^4 + \frac{1}{4} \left\| v(t - \tau) \right\|_{L_4}^4,
\]

\[
F_\tau^*(v,t) = \tau \sum_{s \in R_\tau(t-\tau)} \left( \bar{f}(s), \bar{D}_\tau v(s) \right).
\]

By (4.15), we get that

\[
E_\tau^*(u_N,t) = E_\tau^*(u_N,\tau) + 2F_\tau^*(u_N,\tau) . \tag{4.17}
\]

Clearly it is a reasonable analogy of (4.14).

(4.16) is an implicit scheme. Let

\[
a(v,w) = (\partial_\tau v, \partial_\tau w) + \left( 1 + \frac{2}{\tau^2} \right) (v,w),
\]

\[
b(v,w) = \frac{1}{2} (v^3, w),
\]

\[
d(v, g, w) = \frac{1}{2} (g v^2, w) + \frac{1}{2} (g^2 v, w).
\]
Then at each time \( t \in R, \) we have to solve a nonlinear equation
\[
a(u_N(t), \phi) + b(u_N(t), \phi) + d(u_N(t), u_N(t-2\tau), \phi) = (F(t), \phi), \quad \forall \phi \in P^0_N \quad (4.18)
\]
where \( F(t) \) depends only on \( u_N(t-\tau), u_N(t-2\tau) \) and \( \dot{f}(t-\tau) \). It can be verified that for suitably small \( \tau \), \((4.18)\) has a unique solution at all \( t \in R(T) \).

We now discuss the choice of basis. As we know, Legendre polynomials are orthogonal in \( L^2(\Lambda) \). But it is not so for the basis of \( P^0_N \). We could take the basis of \( P^0_N \) as the set of \( \phi_l(x), 2 \leq l \leq N \) where
\[
\phi_l(x) = \begin{cases} 
L_l(x) - L_0(x), & \text{if } l \text{ is even}, \\
L_l(x) - L_{l-1}(x), & \text{if } l \text{ is odd}.
\end{cases}
\]
Unfortunately this basis leads to a linear system with a full matrix whose elements are \( M_{l,m} = (\phi_l, \phi_m) \). Moreover, the number condition of the matrix \( M^* \) with the elements \( M^* = (\partial_x \phi_l, \partial_x \phi_m) \) is very bad. Shen (1994) proposed another basis, i.e.,
\[
\psi_l(x) = d_l \left( L_l(x) - L_{l+2}(x) \right), \quad d_l = \frac{1}{\sqrt{4l + 6}}, \quad 0 \leq l \leq N - 2.
\]
It can be checked that
\[
(\psi_l, \psi_m) = (\psi_m, \psi_l) = \begin{cases} 
2d_l^2 \left( \frac{1}{2l + 1} + \frac{1}{2l + 5} \right), & l = m, \\
-\frac{2d_l d_{l-2}}{2l + 5}, & l = m + 2, \\
0, & \text{otherwise}.
\end{cases}
\]
In particular,
\[
(\partial_x \psi_l, \partial_x \psi_m) = \delta_{l,m}.
\]
Therefore this basis leads to a five-diagonal matrix and an identity matrix. It saves a lot of work in computation and improve the condition number of the related matrix, which in turn strengthens the stability of algorithms.

We now consider the generalized stability of \((4.16)\). Suppose that \( u_{N,0}, u_{N,1} \) and \( P_N f \) have the errors \( \bar{u}_0, \bar{u}_1 \) and \( \bar{f} \) respectively, which induce the error of \( u_N \), denoted by \( \bar{u}_N \) or \( \tilde{u} \) for simplicity. Then
\[
\begin{align*}
&\left\{ \begin{array}{l}
\tilde{D}_T \bar{u}(t) + \dot{\bar{u}}(t) + \bar{G}(t, \phi) + \left( \partial_x \tilde{u}(t), \partial_x \phi \right) = \left( \bar{f}(t), \phi \right), \quad \forall \phi \in P^0_N, \ t \in R(T), \\
\bar{D}_x \bar{u}(0) = \bar{u}_0, \\
\bar{u}(0) = \bar{u}_0
\end{array} \right.
\end{align*}
\]
where
\[
\bar{G}(x,t) = G(u_N(x,t) + \bar{u}(x,t)) - G(u_N(x,t)) = G(\bar{u}(x,t)) + \bar{R}(x,t),
\]
\[
\left\| \bar{R}(t) \right\|^2 \leq d(u_N) \left( \left\| \bar{u}(t + \tau) \right\|^2 + \left\| \bar{u}(t - \tau) \right\|^2 + \left\| \bar{u}(t + \tau) \right\|^4 + \left\| \bar{u}(t - \tau) \right\|^4 \right) + \sum_{s \in R(T)} \left\| \bar{f}(s) \right\|^2.
\]
According to a priori estimate, \( d(u_N) \) is bounded. Now, let
\[
E(v; t) = \left\| \tilde{D}_T v(t) \right\|^2 + \left\| v(t) \right\|^2 + \left\| v(t) \right\|^4, \\
\rho(t) = (1 + \tau d(u_{N,0})) \left( \left\| \bar{u}_0 \right\|^2 + \left\| \bar{u}_1 \right\|^2 + \left\| \bar{u}_0 \right\|^4 + \left\| \bar{u}_1 \right\|^4 + \tau^4 \right) + 2\tau \sum_{s \in R(t)} \left\| \bar{f}(s) \right\|^2.
\]
We have the following result.
Theorem 4.5. Let }q = \tau N^2 < \infty\text{ and } N \text{ be suitably large. Then for all } t \in \tilde{T}_r(T),
\begin{align*}
E(\hat{u}, t) & \leq c \rho(t) e^{b t}
\end{align*}
where }b_1\text{ is a positive constant depending only on } \|u_N\|_{C(0,T;H^1)}.

Scheme (4.16) possesses the index of generalized stability }s = -\infty\text{.

We next discuss the convergence of (4.16). For the Fourier approximation, the }L^2\text{-orthogonal projection is also the best approximation to the norm } \| \cdot \|_1\text{. But it is not true for Legendre approximation. If we compare the numerical solution } u_N \text{ to } P_N U, \text{ then the leading term in the error equation is } (\partial_x (U - P_N U), \partial_x \phi)\text{. It does not vanish, since } \partial_x P_N \neq P_N \partial_x\text{. This term lowers the convergence rate essentially. Thus, in order to derive the optimal error estimate, we should compare } u_N \text{ to } P_N^{1,0} U\text{. Let } U_N = P_N^{1,0} U \text{ and } \hat{U} = u_N - U_N\text{. Then from (4.13) and (4.16),}
\begin{align*}
\left\{ \begin{array}{l}
\hat{D}_{xx} \hat{U}(t) + \hat{U}(t) + G \left( U_N(t) + \hat{U}(t) \right) - G(U_N(t), \phi) \\
\quad + \left( \partial_x \hat{U}(t), \partial_x \phi \right) = -\sum_{j=1}^{4} (G_j(t), \phi), \quad \forall \phi \in W^0_N, \ t \in R_r(T), \\
D_x \hat{U}(0) = P_N \left( U_1 + \frac{1}{2} \partial^2_x U(0) \right) - P_N^{1,0} \left( U_1 + \frac{1}{2} \partial^2_x U(0) \right) - G_5, \\
\hat{U}(0) = P_N U_0 - P_N^{1,0} U_0
\end{array} \right.
\end{align*}
where
\begin{align*}
G_1(x,t) &= \hat{D}_{xx} U_N(x,t) - \partial^2_x \hat{U}(x,t), \\
G_2(x,t) &= \hat{U}_N(x,t) - \hat{U}(x,t), \\
G_3(x,t) &= G(U_N(x,t)) - G(U(x,t)), \\
G_4(x,t) &= G(U(x,t)) - \frac{1}{2} U^3(x,t + \tau) - \frac{1}{2} U^3(x,t - \tau), \\
G_5(x) &= D_x U_N(x,0) - \partial_x U_N(x,0) - \frac{\tau}{2} \partial^2_x U_N(x,0).
\end{align*}

We have the following conclusion.

Theorem 4.6. Let }q N^2 < \infty \text{ and } r \geq 1\text{. If } U \in C^2(0,T;H^r \cap H^r(0,T;L^2)), U_0 \in H^{r + \frac{1}{2}}(\Lambda) \text{ and } U_1 \in H^r(\Lambda), \text{ then for all } t \in \tilde{T}_r(T),
\begin{align*}
E(U_N - u_N,t) & \leq b_2 e^{b_3 t} \left( \tau^4 + \tau^2 N^2 + \tau^4 N^{-2} + \tau^4 N^{2-r} + N^{-2r} \right)
\end{align*}
where }b_2\text{ and } b_3\text{ are positive constants depending on the norms of } U \text{ and } f \text{ in the spaces mentioned in the above.}

If we take }u_{N,0} = P_N^{1,0} U_0, \text{ and
\begin{align*}
u_{N,1} &= P_N^{1,0} U_1 + \frac{\tau}{2} P_N^{1,0} \left( \partial^2_x U_0 - U_0 - U^3_0 + f(0) \right),
\end{align*}
then we can remove the restrictions on }U_0 \text{ and } U_1 \text{ in Theorem 4.6, and } E(U_N - u_N,t) = \mathcal{O}(\tau^4 + N^{-2r}).
Legendre pseudospectral methods are much easier to be implemented. Let \( x^{(j)} \) and \( \omega^{(j)} \) be the Legendre-Gauss-Lobatto interpolation points and weights. Denote by \( \Lambda_N \) the set of \( x^{(j)}, 0 \leq j \leq N \). We introduce the discrete \( L^p \)-norm associated with the interpolation points as

\[
\|v\|_{L^p, N} = \begin{cases} 
\left( \sum_{j=0}^{N} |v(x_j)|^p \omega^{(j)} \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\
\max_{x \in \Lambda_N} |v(x)|, & \text{for } p = \infty.
\end{cases}
\]

Also define the discrete \( L^2 \)-inner product by

\[
(v, w)_N = \sum_{j=0}^{N} v(x^{(j)}) w(x^{(j)}) \omega^{(j)}.
\]

Let \( I_N v \) be the Legendre interpolant of \( v \in C(\bar{\Lambda}) \), related to the Legendre-Gauss-Lobatto interpolation points. It is easy to see that

\[
\left( G(v(t)), \hat{D}_r v(t) \right)_N = \frac{1}{4} \hat{D}_r \|v(t)\|_{L^4, N}^4. \tag{4.19}
\]

Now, we construct the Legendre pseudospectral scheme for (4.13). Let \( u_N \) approximate \( U \), and \( I_N G(u_N) \) approximate \( U^3 \) instead of \( I_N u_N^3 \). The scheme is to find \( u_N(x, t) \in \mathbb{P}_N^0 \) for all \( t \in \hat{R}_T \) such that

\[
\begin{cases}
\left( \hat{D}_r u_N(t) + \hat{u}_N(t) + G(u_N(t)), \phi \right)_N \\
+ (\partial_x u_N(t), \partial_x \phi) = \left( \hat{f}(t), \phi \right)_N, \quad \forall \phi \in \mathbb{P}_N^0, \ t \in R_T(T), \\
\hat{D}_r u_N(0) = u_{N,1}, \\
u_N(0) = u_{N,0}
\end{cases}
\]

where \( u_{N,0} = I_N U_0 \) and

\[
u_{N,1} = I_N U_1 + \frac{\tau}{2} I_N (\partial_x^2 U_0 - U_0 - U_0^3 + f(0)).
\]

We next check the conservation. Let

\[
E^*_r, N (v, t) = \frac{1}{4} \|\hat{D}_r v(t)\|_{L^4, N}^4 + \frac{1}{2} \|v(t)\|_{L^2, N}^2 + \frac{1}{2} \|v(t - \tau)\|_{L^2, N}^2
\]

\[
+ \frac{1}{4} \|v(t)\|_{L^4, N}^4 + \frac{1}{4} \|v(t - \tau)\|_{L^4, N}^4 + \frac{1}{4} \|v(t)\|_{L^4, N}^4 + \frac{1}{4} \|v(t - \tau)\|_{L^4, N}^4.
\]

We derive from (4.19) and (4.20) that

\[
E^*_r, N (u_N, t) = E^*_r, N (u_N, \tau) + 2\tau \sum_{s \in R_T(1-\tau)} \left( \hat{f}(s), \hat{D}_r u_N(s) \right)_N
\]

which is also a reasonable analogy of (4.14).
For analyzing the generalized stability of (4.20), let \( u_{N,0}, u_{N,1} \) and \( I_N \tilde{f} \) have the errors \( \tilde{u}_0, \tilde{u}_1 \) and \( \tilde{f} \) respectively, which cause the error of \( u_N \), denoted by \( \tilde{u} \). Then

\[
\begin{align*}
\left\{ \begin{array}{l}
(\tilde{D}_\tau \tilde{u}(t) + \tilde{\tilde{u}}(t) + \tilde{G}(t), \phi) + (\dot{\partial}_\phi \tilde{u}(t), \partial_x \phi) = (\tilde{f}(t), \phi), \quad \forall \phi \in \mathbb{P}_N^0, \quad t \in R_\tau(T), \\
\tilde{D}_\tau \tilde{u}(0) = \tilde{u}_1,
\end{array} \right.
\end{align*}
\]

where

\[
\tilde{G}(x, t) = G(\tilde{\tilde{u}}(x, t)) + \tilde{R}(x, t)
\]

and

\[
\left\| \tilde{R}(t) \right\|_N^2 \leq d(u_N) \left( \left\| \partial_x \tilde{\tilde{u}}(t) \right\|_N^2 + \left\| \partial_x \tilde{u}(t - \tau) \right\|_N^2 \\
+ \left\| \tilde{\tilde{u}}(t + \tau) \right\|_N^2 + \left\| \tilde{u}(t - \tau) \right\|_N^2 \right).
\]

It can be shown that \( d(u_N) \) is bounded. Let

\[
E_N(v, t) = \| \tilde{D}_\tau v(t) \|_N^2 + |v(t)|_1^2 + \| v(t) \|_{L^4}^4,
\]

\[
\rho(t) = (c + \tau d(u_{N,0})) \left( \| \tilde{u}_0 \|_1^2 + \| \tilde{u}_1 \|_1^2 + \left( \frac{9}{4} + \tau d(u_{N,0}) \right) \| \tilde{u}_0 \|_{L^4}^4 \right. \\
+ 2\tau^4 \| \tilde{u}_1 \|_{L^4}^4 + 2\tau \sum_{s \in R_\tau(t)} \| \tilde{f}(s) \|_2^2.
\]

We have the following result.

**Theorem 4.7.** Let \( q = \tau N^2 < \infty \) and \( N \) be suitably large. Then for all \( t \in R_\tau(T) \),

\[
E_N(\tilde{u}, t) \leq c \rho(t) e^{b_4 t},
\]

\( b_4 \) being a positive constant depending only on \( \| u_N \|_{C(0,T;H^1)} \).

We now consider the convergence. We could compare \( u_N \) with \( I_N \tilde{U} \). In this case, the optimal error estimate fails, since \( I_N \partial_x^2 \neq \partial_x^2 I_N \). So we compare \( u_N \) with \( P_N^1 \tilde{U} \).

Put \( U_N = P_N^1 \tilde{U} \) and \( \tilde{U} = u_N - U_N \). In this case, (4.13) and (4.20) yield that

\[
\begin{align*}
\left\{ \begin{array}{l}
(\tilde{D}_\tau \tilde{U}(t) + \tilde{U}(t) + G(U_N(t) + \tilde{U}(t)) - G(U_N(t), \phi) \right)_N \\
+ (\partial_\phi \tilde{U}(t), \partial_x \phi) + \sum_{j=1}^4 G_j(t) = 0, \quad \forall \phi \in \mathbb{P}_N^0, \quad t \in R_\tau(T), \\
\tilde{D}_\tau \tilde{U}(0) = I_N \left( U_1 + \frac{1}{\tau} \partial_\phi^2 U_0(0) \right) - P_N^1 \left( U_1 + \frac{1}{\tau} \partial_\phi^2 U(0) \right) - G_5, \\
\tilde{U}(0) = I_N U_0 - P_N^1 \tilde{U},
\end{array} \right.
\end{align*}
\]

where

\[
G_1(t) = (\tilde{D}_\tau U_N(t), \phi)_N - (\partial_\phi^2 \tilde{U}(t), \phi),
\]
\[ G_2(t) = \left( \hat{U}_N(t), \phi \right)_N - \left( \hat{U}(t), \phi \right), \]
\[ G_3(t) = (G (U_N(t)), \phi)_N - \frac{1}{2} (U^3(t + \tau) + U^3(t - \tau), \phi), \]
\[ G_4(t) = \left( f(t), \phi \right) - \left( \hat{f}(t), \phi \right)_N, \]
\[ G_5(x) = D_x U_N(x, 0) - \partial_t U_N(x, 0) - \frac{\tau}{2} \partial_x^2 U_N(x, 0). \]

Comparing this error equation to (4.21), it suffices to deal with \( G_j(t) \). Finally we get the following convergence result.

**Theorem 4.8.** Let \( \tau N^2 < \infty \) and \( r \geq 1 \). If \( U \in C^2(0,T;H^r) \cap C^3(0,T;C) \cap H^4(0,T;L^2), U_0 \in H^{r+1}(\Lambda), U_1 \in H^r(\Lambda) \) and \( f \in L^2(0,T;H^r) \), then for all \( t \in \bar{R}_e(T) \),
\[ E_N(U_N - u_N, t) \leq b \tau b \tau t \left( \tau^4 + \tau^{12} N^2 + \tau^8 N^{-2} + \tau^4 N^{2-r} + N^{-2r} \right), \]
\( b \tau \) and \( b \) being positive constants depending on the norms of \( U, U_0 \) and \( f \) in the spaces mentioned above.

**4.3. Chebyshev Spectral Methods And Chebyshev Pseudospectral Methods**

Chebyshev spectral methods and Chebyshev pseudospectral methods are also used widely for non-periodic problems. Since Chebyshev polynomials can be changed into trigonometric polynomials by a transformation of independent variable, the Fast Fourier Transformation can be used in actual computations. We take the Burgers’ equation as an example to show how to build Chebyshev spectral schemes and Chebyshev pseudospectral scheme.

Burgers’ equation is one of the important models describing the movement of unsteady one-dimensional fluid flow. Let \( \Lambda = (-1, 1) \). The initial-boundary value problem of Burgers’ equation is the following

\[
\begin{aligned}
\{ \quad & \partial_t U(x,t) + U(x,t) \partial_x U(x,t) - \mu \partial_x^2 U(x,t) = f(x,t), \quad x \in \Lambda, \quad 0 < t \leq T, \\
& U(-1, t) = U(1, t) = 0, \quad 0 < t \leq T, \\
& U(x, 0) = U_0(x), \quad x \in \Lambda
\end{aligned}
\]  
(4.22)

where \( U_0(x), f(x,t) \) and \( \mu > 0 \) are the initial state, the body force and the kinetic viscosity respectively. It is known that if \( U_0 \in L^2(\Lambda) \) and \( f \in L^2(0,T;L^2) \), then (4.22) has a unique solution \( U \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2) \). The solution of (4.22) possesses the conservation

\[
\|U(t)\|^2 + 2\mu \int_0^t \|U(s)\|^2 ds = \|U_0\|^2 + 2 \int_0^t (f(s), U(s)) ds. \tag{4.23}
\]

Let \( \omega(x) = \left( 1 - x^2 \right)^{-\frac{1}{2}} \). The space \( L^p_\omega(\Lambda) \) is defined as before, equipped with the inner product \( \langle \cdot, \cdot \rangle_\omega \) and the norm \( \| \cdot \|_\omega \). The spaces \( L^p_\omega(\Lambda), H^m_\omega(\Lambda), H^m_0(\Lambda) \) and their norms \( \| \cdot \|_{L^p_\omega}, \| \cdot \|_{H^m_\omega} \) are also the same as before. Let \( N \) be any positive integer
and $P_N$ be the $L^2$-orthogonal projection from $L^2_\omega(\Lambda)$ onto $\mathbb{P}_N^0$. The three different $H^1_\omega$-orthogonal projections are $P_N^1$, $P_N^1$, and $P_N^{1,0}$. The mesh size in time $t$ is taken to be $\tau$. The notations $R_\tau(T)$, $\hat{v}(x, t)$ and $\hat{D}_\tau v(x, t)$ are the same as before.

When we approach (4.22) by the Legendre spectral approximations, we can get a scheme whose solution fulfills a conservation simulating (4.23). But this property is destroyed by the weight function in Chebyshev spectral approximations. In the previous parts, we constructed the schemes with the accuracy of first order in time discretization, unless the nonlinear term is approximated by a fully implicit technique. If we use three-level scheme, then we can obtain the accuracy of second order in temporal discretization. We shall adopt this trick.

Let $u_N$ be the approximation to $\hat{U}$. A Chebyshev spectral scheme for (4.22) is to find $u_N(x, t) \in \mathbb{P}_N^0$ for all $t \in \bar{R}_\tau(T)$ such that

\[
\begin{cases}
\left( \hat{D}_\tau u_N(t), \phi \right)_\omega - \frac{1}{\mu} \left( u_N^2(t), \partial_x (\phi \omega) \right)_\omega \\
+ \mu \left( \partial_x u_N(t), \partial_x (\phi \omega) \right)_\omega = \left( \hat{f}(t), \phi \right)_\omega, \quad \forall \phi \in \mathbb{P}_N^0, \ t \in R_\tau(T),
\end{cases}
\]

(4.24)

where $u_{N,0} = P_N U_0$ and

\[
u_{N,1} = P_N (U_0 + \tau \partial_t U(0)) = P_N U_0 + \tau P_N \left( \mu \partial_x^2 U_0 + \frac{1}{2} \partial_x U_0^2 + f(0) \right).
\]

At each time $t \in R_\tau(T)$, we need to solve a linear system. It possesses a unique solution.

In actual calculations, we can take the base functions as

\[
\phi_l(x) = \begin{cases}
T_i(x) - T_0(x), & l \text{ is even}, \\
T_i(x) - T_1(x), & l \text{ is odd}.
\end{cases}
\]

But these functions lead to a linear system with a full matrix. A better choice (see Shen (1995)) is to take

\[
\psi_l(x) = T_l(x) - T_{l+2}(x).
\]

Then

\[
(\psi_m, \psi_l)_\omega = (\psi_l, \psi_m)_\omega = \begin{cases}
\frac{\pi}{2} (c_m + 1), & l = m, \\
n\pi, & l = m - 2 \text{ or } m + 2, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
(\partial_x \psi_l, \partial_x (\psi_m \omega)) = \begin{cases}
2\pi (m + 1)(m + 2), & l = m, \\
4\pi (m + 1), & l = m + 2, m + 4, \ldots, \\
0, & l < m \text{ or } l + m \text{ odd}.
\end{cases}
\]

We now consider the generalized stability of (4.24). Let $\tilde{u}_0, \tilde{u}_1$ and $\tilde{f}$ be the errors of $u_{N,0}, u_{N,1}$ and $P_N \hat{f}$, which induce the error of $u_N$, say $\tilde{u}$. Set

\[
E(v, t) = ||v(t)||_\omega^2 + \frac{1}{2} \mu \tau \sum_{s \in R_\tau(t-\tau)} ||\tilde{u}(s)||^2_{1, \omega},
\]
\[
\rho(t) = 2 \|\tilde{u}_0\|_\omega^2 + 2 \|\tilde{u}_1\|_\omega^2 + 4\tau \sum_{s \in \tilde{R}_r(t)} \|f(s)\|_\omega^2.
\]

We have the following result.

**Theorem 4.9.** Let \(\tau\) be suitably small. There exist positive constants \(b_1\) and \(b_2\), depending only on \(\|u_N\|_{C(0,T,L^2)}\) and \(\mu\), such that if \(\rho(t_1) \leq \frac{b_1}{N}\) for certain \(t_1 \in R_r(T)\), then for all \(t \in \tilde{R}_r(t_1)\),

\[
E(\tilde{u}, t) \leq \rho(t)e^{b_2t}.
\]

We now deal with the convergence. We can compare \(u_N\) to \(P_NU, P_N^1U\) or \(\tilde{P}_N^1U\). But in these cases, the leading error terms cannot be cancelled. It means that there will exist the terms \((\partial_x(U - P_NU), \partial_x(\phi\omega))\), \((\partial_x(U - P_N^1U), \partial_x(\phi\omega))\), or \((\partial_x(U - \tilde{P}_N^1U), \partial_x(\phi\omega))\). All of them do not vanish for \(\phi \in \mathbb{P}_N^0\). They shall lower the convergence rate. Therefore we compare \(u_N\) to \(U_N = P_N^1U\), where the \(H^1\)-projection \(P_N^1\) is defined as in (2.22). The following result exists.

**Theorem 4.10.** Let \(\tau \leq b_3N^{-4}\). If \(U \in C^1(0,T;H^1_{0,\omega} \cap H^3(0,T;L^2_{\omega}))\) with \(r \geq 1\), then

\[
E(U_N - u_N, t) \leq b_4e^{b_5t}(\tau^4 + N^{-2r}),
\]

\(b_3, b_4\) and \(b_5\) being positive constants depending on \(\mu\) and the norms of \(U\) in the spaces mentioned in the above.

Chebyshev pseudospectral methods are more preferable in practical cases. Let \(x^{(j)}\) and \(\omega^{(j)}\) be the Chebyshev-Gauss-Lobatto interpolation points and weights. Set \(\Lambda_N = \{x^{(j)} | 0 \leq j \leq N\}\). Denote by \(I_{N\mu}\) the Chebyshev interpolant of \(v \in C(\Lambda)\), associated with \(\Lambda_N\). The discrete inner product \((\cdot, \cdot)_{N,\omega}\) and the discrete norm \(\|\cdot\|_{N,\omega}\) are defined by (2.10), related to the Chebyshev-Gauss-Lobatto points and weights.

Let \(u_N\) be the approximation to \(U\). In order to strengthen the stability, we use the filtering \(R_N = R_N(\alpha, \beta), \alpha, \beta \geq 1\), defined by (2.36). The Chebyshev pseudospectral scheme for (4.22) is to find \(u_N(x,t) \in \mathbb{P}_N^0\) for all \(t \in \tilde{R}_r(T)\) such that

\[
\begin{cases}
(D_x u_N(t), \phi)_{N,\omega} - \frac{1}{\mu}(R_N I_N u^2_N(t), \partial_x(\phi\omega)) \\
\quad + \mu(\partial_x \tilde{u}_N(t), \partial_x(\phi\omega)) = (R_N I_N \tilde{f}(t), \phi)_{N,\omega}, \quad \forall \phi \in \mathbb{P}_N^0, t \in R_r(T),
\end{cases}
\]

\[
\begin{align*}
&u_N(\tau) = u_{N,1}, \\
&u_N(0) = u_{N,0}
\end{align*}
\]

where \(u_{N,0} = I_N U_0\) and

\[
u_{N,1} = I_N (U_0 + \tau \partial_t U(0)) = I_N U_0 + \tau I_N \left(\mu \partial_x^2 U_0 - \frac{1}{2} \partial_x U_0^2 + f(0)\right).
\]

Now suppose that \(u_{N,0}, u_{N,1}\) and \(R_N I_N \tilde{f}\) have the errors \(\tilde{u}_0, \tilde{u}_1\) and \(\tilde{f}\), which cause \(\tilde{u}_N\), the error of \(u_N\). Let \(E(v, t)\) be the same as before, and

\[
\rho(t) = c \left(\|\tilde{u}_0\|_\omega^2 + \|\tilde{u}_1\|_\omega^2 + \tau \sum_{s \in \tilde{R}_r(t)} \|f(s)\|_\omega^2\right).
\]
Theorem 4.11. Let $\tau$ be suitably small. There exist positive constants $b_0$ and $b_1$ depending only on $\|u_N\|_{C(0,T;L^\infty)}$ and $\mu$, such that if $\rho(t_1) \leq \frac{b_0}{N}$ for some $t_1 \in R_r(T)$, then for all $t \in R_r(t_1)$,

$$E(\bar{u}(t)) \leq \rho(t)e^{b_1t}.$$  

We next state the convergence of (4.25). For the optimal error estimate, we also compare $u_N$ to $U_N = P_N^{\frac{b_0}{b_1}}U$. We have the following result.

Theorem 4.12. Let $\tau \leq b_0N^{-\frac{1}{2}}$. If $U \in C^1(0,T; H^1_0(\omega) \cap H_0^2(\omega)) \cap H^3(0,T; L^2_N), f \in C(0,T; H^1_0(\omega))$ and $1 \leq r \leq \alpha$, then

$$E(U_N - u_N(t)) \leq b_0e^{b_1t}\left(\tau^4 + N^{-2r}\right),$$  

$b_0, b_1$ and $b_{10}$ being positive constants depending on $\mu$ and the norms of $U$ and $f$ in the spaces mentioned in the above.

4.4. Spectral Penalty Methods

We can deal with inhomogeneous boundary conditions in several ways. One of them is the spectral penalty method in which the boundary conditions become a part of equation. We can use Chebyshev penalty method as described below. Consequently, we adopt the weighted norm in numerical analysis. However, this is not a natural norm and complicates the analysis. A more natural way is to use Legendre-Gauss-Lobatto interpolation and the corresponding penalty method. As those points are not given explicitly, their evaluation for large $N$ is not robust due to roundoff errors. In this part, we focus on a new approach, implementing Legendre method on Chebyshev points, enjoying the advantages of both Legendre and Chebyshev methods. It is named the Chebyshev-Legendre method by Don and Gottlieb (1994).

We begin with a simple problem. Let $\Lambda = (-1,1)$ and consider the hyperbolic equation

$$\begin{cases}
\partial_t U(x,t) - \partial_x U(x,t) = f(x,t), & x \in \Lambda, t > 0, \\
U(1,t) = g(t), & t > 0, \\
U(x,0) = U_0(x), & x \in \Lambda.
\end{cases}$$  

(4.26)

Let $x^{(j)}$ and $\omega^{(j)}$ be the Chebyshev-Gauss-Lobatto interpolation points and weights, $\Lambda_N = \{x^{(j)} \mid 0 \leq j \leq N\}$. The usual semi-discrete Chebyshev pseudospectral scheme is to find $u_N(x,t) \in P_N$ for $t \geq 0$ such that

$$\begin{cases}
\partial_t u_N(x,t) - \partial_x u_N(x,t) = f(x,t), & x \in \Lambda_N \setminus \{1\}, t > 0, \\
u_N(1,t) = g(t), & t > 0, \\
u_N(x,0) = U_0(x), & x \in \Lambda_N.
\end{cases}$$  

(4.27)

Funaro and Gottlieb (1988) proposed a penalty method in which, instead of the boundary condition in (4.27), we require that

$$\partial_t u_N(1,t) - \partial_x u_N(1,t) + \lambda (u_N(1,t) - g(t)) = f(1,t)$$

where $\lambda$ is determined from stability considerations. This approach is stable for the initial data as long as $\lambda \geq \frac{1}{2}N^2$. Since $x^{(j)}$ are the zeros of the polynomial
\((1 - x^2) \partial_x T_N(x)\), this scheme can be written as

\[
\begin{cases}
\frac{\partial}{\partial t} u_N(x, t) - \partial_x u_N(x, t) + \frac{\lambda(1 + x) \partial_x T_N(x)}{2 \partial_x T_N(1)} \left( u_N(1, t) - g(t) \right) = f(x, t), & x \in \tilde{\mathcal{X}}, \ t > 0, \\
u_N(x, 0) = U_0(x), & x \in \tilde{\mathcal{X}}.
\end{cases}
\tag{4.28}
\]

The main feature of (4.28) is that the numerical solution does not fulfill the boundary condition exactly, but only in the limit as \(N \to \infty\). In fact, the boundary condition is a part of the equation. Another penalty method based on Legendre-Gauss-Lobatto interpolation was served by Funaro and Gottlieb (1991). Let \(\tilde{x}^{(j)}\) and \(\tilde{\omega}^{(j)}\) be the nodes and weights of this interpolation. \(\tilde{\mathcal{X}} = \{y^{(j)} = \tilde{x}^{(N-j)} \mid 0 \leq j \leq N\}\). Since \(y^{(j)}\) are the zeros of the polynomial \((1 - x^2) \partial_x L_N(x)\), the penalty scheme for (4.26) is to find \(u_N(x, t) \in \mathbb{P}_N\) for \(t \geq 0\) such that

\[
\begin{cases}
\frac{\partial}{\partial t} u_N(x, t) - \partial_x u_N(x, t) + \frac{\lambda(1 + x) \partial_x L_N(x)}{2 \partial_x L_N(1)} \left( u_N(1, t) - g(t) \right) = f(x, t), & x \in \tilde{\mathcal{X}}, \ t > 0, \\
u_N(x, 0) = U_0(x), & x \in \tilde{\mathcal{X}}.
\end{cases}
\tag{4.29}
\]

This scheme is also stable for the initial data when \(\lambda \geq \frac{1}{2} N(N + 1)\).

The Chebyshev-Legendre penalty scheme for (4.26) is to find \(u_N(x, t) \in \mathbb{P}_N\) for \(t \geq 0\) such that

\[
\begin{cases}
\frac{\partial}{\partial t} u_N(x, t) - \partial_x u_N(x, t) + \frac{\lambda(1 + x) \partial_x L_N(x)}{2 \partial_x L_N(1)} \left( u_N(1, t) - g(t) \right) = f(x, t), & x \in \tilde{\mathcal{X}}, \ t > 0, \\
u_N(x, 0) = U_0(x), & x \in \tilde{\mathcal{X}}.
\end{cases}
\tag{4.30}
\]

We now explore the relation between (4.29) and (4.30). Let

\[
q_C(x) = (1 - x^2) \partial_x T_N(x), \\
q_L(x) = (1 - x^2) \partial_x L_N(x).
\]

Define the Lagrange polynomials

\[
g_l(x) = \frac{q_C(x)}{(x - x^{(l)}) \partial_x q_C(x^{(l)})}, & 0 \leq l \leq N, \\
h_l(x) = \frac{q_L(x)}{(x - y^{(l)}) \partial_x q_L(y^{(l)})}, & 0 \leq l \leq N.
\]

Denote by \(D_C\) the differentiation matrix in Chebyshev pseudospectral approximation, and by \(D_L\) the differentiation matrix in Legendre pseudospectral approximation. Let \(A\) and \(B\) be the \((N + 1) \times (N + 1)\) matrices with the elements

\[
A_{l,m} = h_m \left( x^{(l)} \right), \quad B_{l,m} = g_m \left( y^{(l)} \right), & 0 \leq l, m \leq N.
\]

Furthermore, denote by \(F\) the matrix induced by the differentiation and the penalty in (4.30), and denote by \(G\) the matrix induced by the differentiation and the penalty in (4.29). Then \(AB = I, D_C = AD_L B\) and \(F = AG B\).

We now turn to analyze the stability of scheme (4.30). It suffices to verify that the norm of \(u_N\) depends on the norms of data continuously.
THEOREM 4.13. If \( \frac{1}{4}(1 + \varepsilon)(N^2 + N) \leq \lambda \leq c_1(N^2 + N) \) and \( \varepsilon > 0 \), then
\[
\|u_N(t)\|_N^2 \leq e^{\varepsilon t} \left( \|u_N(0)\|_N^2 + \frac{1}{\varepsilon} \int_0^t \left( \|I_N f(s)\|_N^2 + 4c_1^2 g^2(s) \right) ds \right).
\]

Theorem 4.13 shows the linear stability of scheme (4.30). In particular, if \( f(x, t) = g(t) = 0 \), then
\[
\|u_N(t)\|_N^2 + \int_0^t \left( \frac{4\lambda}{N(N+1)} - 1 \right) u_N^2(1, s) + u_N^2(-1, s) \right) ds = \|u_N(0)\|_N^2.
\]

Next, we consider the convergence of (4.30). Let \( U_N = I_N U \). We get from (4.26) that
\[
\begin{align*}
\partial_t U_N(x, t) - \partial_x U_N(x, t) + \frac{\lambda(1 + x)\partial_x L_N(x)}{2\partial_x L_N(1)}(U_N(1, t) - g(t)) \\
= G_1(x, t) + G_2(x, t) + I_N f(x, t), \quad x \in \bar{\Lambda}, \ t > 0, \\
U_N(x, 0) = I_N U_0(x) + G_3(x), \quad x \in \bar{\Lambda}
\end{align*}
\]
where
\[
G_1(x, t) = I_N \partial_x U(x, t) - \partial_x I_N U(x, t),
G_2(x, t) = I_N f(x, t) - I_N f(x, t),
G_3(x, t) = I_N U_0(x) - I_N U_0(x).
\]

Put \( \bar{U} = u_N - U_N \). Then for all \( x \in \bar{\Lambda} \) and \( t > 0 \),
\[
\partial_t \bar{U}(x, t) - \partial_x \bar{U}(x, t) + \frac{\lambda(1 + x)\partial_x L_N(x)}{2\partial_x L_N(1)} \bar{U}(1, t) = -G_1(x, t) - G_2(x, t).
\]

We have the following result.

THEOREM 4.14. Let \( \lambda \geq \frac{1}{4}N(N+1) \) and \( r > \frac{1}{2} \). If \( U \in L^2(0, T; H^r+1) \), \( U_0 \in H^s_w(\Lambda) \) and \( f \in L^2(0, T; H^r_w) \), then for all \( t \leq T \),
\[
\|U_N - U_N\| \leq b_1 N^{-r}
\]
where \( b_1 \) is a positive constant depending only on the norms of \( U, U_0 \) and \( f \) in the mentioned spaces.

Now we apply the Chebyshev-Legendre penalty method to parabolic equations. Consider the problem
\[
\begin{align*}
\partial_t U(x, t) - \partial_{xxxx} U(x, t) &= f(x, t), \quad x \in \Lambda, \ t > 0, \\
\alpha\partial_x U(1, t) + \beta U(1, t) &= g^+(t), \quad t > 0, \\
-\gamma\partial_x U(-1, t) + \delta U(-1, t) &= g^-(t), \quad t > 0, \\
U(x, 0) &= U_0(x), \quad x \in \bar{\Lambda}
\end{align*}
\]
(4.31)

where \( \alpha, \beta, \gamma, \delta \geq 0, \alpha + \beta > 0 \) and \( \gamma + \delta > 0 \). In order to describe the Chebyshev-Legendre penalty method, we introduce the operator
\[
R(v, x, t) = \lambda_0 Q_0(x) \left( \alpha \partial_x v(1, t) + \beta v(1, t) - g^+(t) \right) \\
+ \lambda N Q_N(x) \left( -\gamma \partial_x v(-1, t) + \delta v(-1, t) - g^-(t) \right)
\]
with
\[ Q_0(x) = \frac{(1 + x) \partial_x L_N(x)}{2 \partial_x L_N(1)}, \quad Q_N(x) = \frac{(1 - x) \partial_x L_N(x)}{2 \partial_x L_N(-1)}. \]

The Chebyshev-Legendre penalty scheme for (4.31) is to seek \( u_N(x, t) \in \mathcal{P}_N \) for \( t \geq 0 \) such that
\[
\begin{cases}
\partial_t u_N(x, t) - \partial_x^2 u_N(x, t) + R(u_N, x, t) = f(x, t), & x \in \bar{\Omega}, \ t > 0, \\
u_N(x, 0) = U_0(x), & x \in \bar{\Omega}.
\end{cases}
\] (4.32)

We first consider the stability. Let \( \kappa = \frac{b \gamma(0)}{a} \) and
\[
c(a, b) = \frac{1}{a \omega(0)} \left( 1 + 2k + 2\sqrt{k^2 + k} \right),
\]
\[
d(a, b) = \frac{1}{a \omega(0)} \left( 1 + 2k - 2\sqrt{k^2 + k} \right).
\]

**Theorem 4.15.** Let \( d(\alpha, \beta) \leq \lambda_0 \leq c(\alpha, \beta) \) and \( d(\gamma, \delta) \leq \lambda_N \leq c(\gamma, \delta) \). If \( g^+ (t) = g^- (t) = 0 \), then for any \( \varepsilon > 0 \),
\[
\|u_N(t)\|_N^2 + 2 \int_0^t \sum_{j=1}^{N-1} \left( \partial_x u_N \left( y^{[j]}(s), s \right) \right)^2 \omega^{[j]}(s) ds \leq c \varepsilon t \left( \|u_N(0)\|_N^2 + \frac{1}{\varepsilon} \int_0^t \|I_N f(s)\|_N^2 \right) ds.
\]

We next consider the convergence of scheme (4.32). For simplicity let \( \alpha = \gamma = 0, \beta = \delta = 1 \). Let \( P_N^{1,0} \) be the \( H^2 \)-orthogonal projection upon \( \mathcal{P}_N^0 \) defined by (2.17), and \( U_N = P_N^{1,0} U \). Then by (4.31),
\[
\begin{cases}
(\partial_t U_N(t), \phi)_N + (\partial_x U_N(t), \partial_x \phi)_N + (R(U_N, t), \phi)_N \\
= G_1(t) + G_2(t) + (I_N f(t), \phi)_N, & \forall \phi \in \mathcal{P}_N^0, \ t > 0,
U_N(0) = P_N^{1,0} U_0,
\end{cases}
\]

where
\[
G_1(t) = (\partial_t U(t), \phi)_N - (\partial_t U(t), \phi),
\]
\[
G_2(t) = (f(t), \phi) - (f(t), \phi)_N + (f(t) - I_N f(t), \phi)_N.
\]

On the other hand,
\[
(\partial_t U_N(t), \phi)_N + (\partial_x U_N(t), \partial_x \phi)_N + (R(u_N, t), \phi)_N = (I_N f(t), \phi)_N.
\]

Set \( \bar{U} = u_N - U_N \). We obtain that
\[
\begin{cases}
(\partial_t \bar{U}(t), \phi)_N + (\partial_x \bar{U}(t), \partial_x \phi)_N + (R(t), \phi)_N = -G_1(t) - G_2(t), \\
\bar{U}(0) = I_N U_0 - P_N^{1,0} U_0.
\end{cases}
\]

**Theorem 4.16.** Let \( \alpha = \gamma = 0, \beta = \delta = 1, r \geq 1 \) and \( \lambda_0, \lambda_N \geq \frac{1}{16} N^2 (N + 1)^2 \). If \( U \in H^1(0, t; H^r), U_0 \in H^r(\bar{\Omega}) \) and \( f \in L^2(0, T; H^r), \) then for all \( t \leq T, \)
\[
\|u_N(t) - u(t)\| \leq b_2 N^{-r}
\]
where \( b_2 \) is a positive constant depending only on the norms of \( U, U_0 \) and \( f \) in the mentioned spaces.
The Chebyshev-Legendre penalty method can be used for nonlinear problems. Let \( f(z) \) be a continuous convex-function and \( f'(z) = \partial_z f(z) \). Consider the problem
\[
\partial_t U(x, t) + \partial_x f(U(x, t)) - \mu \partial_x^2 U(x, t) = F(x, t), \quad x \in \Lambda, \ 0 < t \leq T
\]
with \( \mu \geq 0 \) and the boundary conditions
\[
U(1, t) = g^+(t), \quad \text{if } \mu > 0 \text{ or } \mu = 0, f'(g^+(t)) \leq 0,
\]
\[
U(-1, t) = g^-(t), \quad \text{if } \mu > 0 \text{ or } \mu = 0, f'(g^-(t)) \geq 0.
\]
The corresponding Chebyshev-Legendre penalty method is as follows
\[
\partial_t u_N(x, t) + \partial_x I_N f(u_N(x, t)) - \mu \partial_x^2 u_N(x, t) + R(u_N, x, t) = F(x, t), \quad x \in \bar{\Lambda}_N, \ t > 0
\]
with
\[
R(v, x, t) = b_0(t)\lambda_0 Q_0(x) \left(v(1, t) - g^+(t)\right) + b_N(t)\lambda_N Q_N(x) \left(v(-1, t) - g^-(t)\right)
\]
where \( b_0(t) = 1 \) for \( \mu > 0 \) or \( \mu = 0, f'(g^+(t)) \leq 0 \), and \( b_0(t) = 0 \) otherwise; and \( b_N(t) = 1 \) for \( \mu > 0 \) or \( \mu = 0, f'(g^-(t)) \geq 0 \), and \( b_N(t) = 0 \) otherwise.

4.5. Spectral Vanishing Viscosity Methods

In this part, we study the spectral methods for nonlinear conservation laws whose solutions may develop spontaneous jump discontinuities, i.e., shock waves. Hence weak solutions must be admitted. There are many weak solutions for the same problem usually. So we have to single out the unique “physically relevant” solution satisfying the entropy condition. We shall describe the spectral vanishing viscosity methods for nonlinear conservation laws such that the numerical solutions converge to weak solution fulfilling the entropy condition. Such solutions are called the entropy solutions.

Let \( \mathbb{R}^1 = \{x \mid -\infty < x < \infty\} \), \( \mathbb{R}_1^+ = \{t \mid t > 0\} \) and \( Q = \mathbb{R}^1 \times \mathbb{R}_1^+ \). Let \( D \) be a set in \( \mathbb{R}^2 \). If \( D_1 \) is a compact set in \( D \), then we say that \( D_1 \subset D \). Moreover, if \( v(x, t) \) is defined on \( D \) almost everywhere, then we say that \( v(x, t) \) is defined on \( D \), a.e.. If a function \( v(x, t) \) is defined on \( D \), a.e., and for any measurable subset \( D_1 \subset D \), \( v \in L^q(D_1), 1 \leq q \leq \infty \), then it is denoted by \( v \in L^q_{\text{loc}}(D) \). We can define the spaces \( H^s_{\text{loc}}(D) \) and \( W^{r, q}_{\text{loc}}(D) \) similarly. Furthermore, if \( v_i \) is a sequence such that
\[
\lim_{i \to \infty} \int_D v_i(x, t) w(x, t) \, dx \, dt = \int_D v(x, t) w(x, t) \, dx \, dt, \quad \forall w \in L^1(D),
\]
then we say that \( v_i \) converges to \( v \) in \( L^\infty(D) \) weak-star. Besides, let \( \mathcal{D}(Q) \) be the space involving all infinitely differentiable functions vanishing for sufficiently large \( |x| + t \).

We now discuss the nonlinear conservation laws. Let \( f(z) \) be a convex function in \( C^1(\mathbb{R}^1) \), and \( f'(z) = \partial_z f(z) \). Consider the problem
\[
\begin{cases}
\partial_t U(x, t) + \partial_x f(U(x, t)) = 0, & (x, t) \in Q, \\
U(x, 0) = U_0(x), & x \in \mathbb{R}^1.
\end{cases}
\] (4.33)
A weak solution of (4.33) is a bounded measurable function $U(x,t)$ such that
\[
\int_Q (U(x,t)\partial_t w(x,t) + f(U(x,t))\partial_x w(x,t)) \, dxdt + \int_{\mathbb{R}^1} U_0(x) w(x,0) \, dx = 0, \quad \forall \, w \in \mathcal{D}(Q).
\]

Next let $E(z)$ be certain smooth function in $\mathbb{R}^1$, and
\[
F(z) = \int E'(z)f'(z) \, dz.
\]

If for any smooth solution of (4.33), there holds
\[
\partial_t E(U(x,t)) + \partial_x F(U(x,t)) = 0, \quad (x,t) \in Q,
\]
then we say that $E$ and $F$ are a pair of entropy and entropy flux for (4.33). If for any strictly convex entropy $E(z)$, a weak solution $U(x,t)$ satisfies
\[
\partial_t E(U(x,t)) + \partial_x F(U(x,t)) \leq 0, \quad (x,t) \in Q, \quad (4.34)
\]
then we say that $U(x,t)$ is an admissible solution. It can be verified that if both $U(x,t)$ and $V(x,t)$ are admissible solutions of (4.33), and
\[
\lim_{t \to 0^+} U(x,t) = \lim_{t \to 0^+} V(x,t), \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^1),
\]
then $U(x,t) = V(x,t)$ in $\bar{Q}$, a.e.

There are fruitful results for the existence of weak solutions, see Lax (1972). Tartar (1975) used compensated compactness to study (4.33). The key point is to explore the conditions with which a sequence $\{u_t\}$ tends to a weak solution as it tends to $v$ in $L^\infty(Q)$ weak-star, or in other words $f(u_t)$ tends to $f(v)$.

We first consider periodic problems. Let $\Lambda = (0,2\pi)$ and $Q = \Lambda \times \mathbb{R}^1_+$. Denote by $\mathcal{D}_p(Q)$ the space involving all infinitely differentiable functions with the period $2\pi$ for the variable $x$, vanishing for sufficiently large $t$. Let $D = \Lambda \times \{t \mid 0 < t_1 < t < t_2\}$. Define the spaces $L^p_p(Q), L^\infty_p(Q)$ and $H^1_p(Q)$ similar. The periodic nonlinear conservation laws are of the form
\[
\begin{cases}
\partial_t U(x,t) + \partial_x f(U(x,t)) = 0, & (x,t) \in Q, \\
U(x,t) = U(x + 2\pi,t), & (x,t) \in Q, \\
U(x,0) = U_0(x), & x \in \Lambda.
\end{cases} \quad (4.35)
\]

The weak solutions of (4.33) are bounded measurable functions such that
\[
\int_Q (U(x,t)\partial_t w(x,t) + f(U(x,t))\partial_x w(x,t)) \, dxdt + \int_{\mathbb{R}^1} U_0(x) w(x,0) \, dx = 0, \quad \forall \, w \in \mathcal{D}_p(Q).
\]

The admissible solutions are also identified by (4.34).

We are going to construct the Fourier spectral scheme. Let $P_N$ be the $L^2$-orthogonal projection upon $V_N$ as defined by (2.2). The standard Fourier spectral scheme is to find $u_N(x,t) \in V_N$ for $t \geq 0$ such that
\[
\begin{cases}
\partial_t u_N(x,t) + \partial_x P_N f(u_N(x,t)) = 0, & (x,t) \in Q, \\
u_N(x,0) = P_N U_0(x), & x \in \Lambda.
\end{cases}
\]
We find after integration that
\[ \| u_N(t) \|^2 = \| u_N(0) \|^2 \leq \| U_0 \|^2. \]
This yields the existence of a weak solution \( U \), which is the weak limit of \( u_N \) in \( L^2(0,T;L^2) \) for any \( t > 0 \). Unfortunately it does not solve our problem. In fact, if \( U \) is a weak solution, then \( P_N f(u_N) \), and hence \( f(u_N) \) should tend to \( f(U) \) weakly.

If in addition \( f(z) = \frac{1}{2} z^2 \), then \( u_N^2 \) tends to \( U^2 \) weakly and so \( u_N \) converges to \( U \) in \( L^2(0,T;L^2) \) strongly. Therefore \( \| U(t) \| \) is also conserved in time. However the inequality (4.34) is valid strictly at the locations of shock waves. It leads to a contradiction. Tadmor (1989) proposed the following Fourier spectral vanishing viscosity scheme
\[
\partial_t u_N(x,t) + \partial_x P_N f(u_N(x,t)) = \varepsilon \partial_x ((I - P_M) \partial_x u_N(x,t)) \tag{4.36}
\]
where \( \varepsilon = \varepsilon(N) \to 0 \) as \( N \to \infty \), and \( M = M(N) < N, M(N) \to \infty \) as \( N \to \infty \).

Now assume that
\[
\varepsilon(N) \sim cN^{-\alpha}, \quad M(N) \sim cN^\beta, \quad 0 < 2\beta < \alpha \leq 1. \tag{4.37}
\]
and
\[
\varepsilon \| \partial_x u_N(0) \|^2 \leq A, \quad \| u_N \|_{L^\infty(0,T;L^\infty)} \leq A. \tag{4.38}
\]
Also assume that there is no interval on which \( f \) is affine. By a compensated compactness argument, Tadmor (1989) proved the following result.

**Theorem 4.17.** Let (4.37) and (4.38) hold. Then for \( 1 \leq q < \infty \), the solution of (4.36) tends to a weak solution of (4.35) in \( L^q_{\text{loc}}(Q) \) strongly. If in addition \( \alpha < 1 \), then it is the unique entropy solution.

In actual computations, we can take the vanishing viscosity term as \( \varepsilon \partial_x (q_N \partial_x u_N) \) where
\[
q_N(x,t) = \sum_{0 \leq |l| \leq N} \hat{q}_l(t) e^{ixl}
\]
and \( \varepsilon = \varepsilon(N) \to 0, M = M(N) \to \infty \) as \( N \to \infty \). For instance, \( M \sim 2N^\beta \) and
\[
\hat{q}_l = \begin{cases} 0, & |l| \leq M, \\ 1, & |l| > M. \end{cases}
\]

The total variations of Fourier viscosity schemes and their stabilities and convergences are discussed in Maday and Tadmor (1989), and Tadmor (1991,1993).

Now, let \( \Lambda = (-1,1), Q = \Lambda \times \mathbb{R}^+_I \) and consider the initial-boundary value problem of (4.33) with appropriate data \( g(t) \in H^1_{\text{loc}}(\mathbb{R}^+_I) \), prescribed at \( x = \pm 1 \). Let \( P_N \) and \( I_N \) be the \( L^2 \)-orthogonal projection upon \( \mathcal{P}_N \) and the interpolation associated with Legendre-Gauss-Lobatto interpolation points \( x^{(j)} \) and weights \( \omega^{(j)} \). Let \( v \in L^2(\Lambda) \) and \( \hat{v} \) be the coefficients of its Legendre expansion. A viscosity operator \( q \) is defined as
\[
qv(x) = \sum_{l=0}^N \hat{q}_l \hat{v}_l L_l(x)
\]
with $M = M(N)$ and
\[
\begin{cases}
  \hat{q}_l = 0, & \text{for } l \leq M, \\
  \hat{q}_l \geq 1 - \frac{M^2}{l^2}, & \text{for } M < l \leq N.
\end{cases}
\]

Let $\varepsilon = \varepsilon(N)$ be the small parameter. The free pair $(\varepsilon, M)$ will be chosen later, such that $\varepsilon \to 0, M \to \infty$ as $N \to \infty$. On the other hand, let
\[
B(u_N(t)) = (\lambda(t)(1 - x) + \mu(t)(1 + x)) \partial_x L_N(x).
\]

The pair $(\lambda, \mu)$ is chosen to match the inflow boundary data prescribed at $x = 1$ whenever $f'(u_N(1, t)) < 0$, and at $x = -1$ whenever $f'(u_N(-1, t)) > 0$. A Legendre vanishing viscosity penalty scheme for the initial-boundary value problem of (4.33) was given by Ma and Guo (1998). It is to find $u_N(x, t) \in \mathbb{P}_N$ for $t \geq 0$ such that

\[
\begin{cases}
  (\partial_t u_N(t) + \partial_x I_N f(u_N(t)), \phi)_N + \varepsilon (\partial_x (q u_N(t)), \partial_x (q \phi))_N \\
  = (B(u_N(t)), \phi)_N, & \forall \phi \in \mathbb{P}_N, t > 0, \\
  u_N(x, t) = g(t), & \text{at the inflow boundary points}, t > 0, \\
  u_N(0, t) = I_N U_0(x), & x \in \Lambda. 
\end{cases}
\]

(4.39)

For simplifying the presentation, we shall deal with the prototype case where one boundary, say $x = -1$, is an inflow boundary, while $x = 1$ is an outflow one. Then
\[
B(u_N(t)) = \lambda(t)(1 - x) \partial_x L_N(x).
\]

Assume that for all $N$,
\[
\varepsilon \|\partial_x u_N(0)\|^2 \leq A, \quad \max_{0 \leq t \leq T} \|u_N(t)\|_{L^\infty(\Lambda)} \leq A. 
\]

(4.40)

and
\[
\varepsilon \sim cN^{-\alpha}, \quad M \sim cN^\beta, \quad 0 < 4\beta < \alpha \leq 1.
\]

(4.41)

**Theorem 4.18.** Let (4.40) and (4.41) hold. Then the solution of (4.39) tends to a weak solution of the initial-boundary value problem of (4.33) in $L^p_{loc}(Q)$ strongly, $1 \leq q < \infty$. If in addition $\alpha < 1$, then it is the unique entropy solution.

Another scheme was proposed by Maday, Kaber and Tadmor (1993).

### 4.6. Spectral Approximations of Isolated Solutions

In the previous parts, we studied various spectral methods for nonlinear revolutionary problems. Generally speaking, all of those methods are applicable to the corresponding steady problems. But for nonlinear elliptic equations, the situations are more complicated. The main feature is that they may possess several solutions. In this part, we take the steady Burgers equation as an example to describe the spectral approximations of isolated solutions.

Let $\Lambda = (-1, 1)$ and $\mu > 0$. $f(x)$ is a given function. Consider the problem
\[
\begin{cases}
  U\partial_x U - \mu \partial_x^2 U = f, & x \in \Lambda, \\
  U = 0, & x = -1, 1.
\end{cases}
\]

(4.42)
Set
\[
\begin{align*}
    a(v, w) &= \int_{\Lambda} \partial_x v(x) \partial_x w(x) \, dx, \quad \forall \, v, w \in H^1(\Lambda), \\
    b(v, w, z) &= \frac{1}{2} \int_{\Lambda} v(x) w(x) \partial_z z \, dx, \quad \forall \, v, w \in L^2(\Lambda), \quad z \in H^1(\Lambda).
\end{align*}
\]

The solution of (4.42) is defined as a function \( U \in H_0^1(\Lambda) \), satisfying
\[
\mu a(U, v) - b(U, U, v) = (f, v), \quad \forall \, v \in H_0^1(\Lambda).
\] (4.43)

If \((f, v)\) is a linear continuous functional in \( H_0^1(\Lambda) \), then by Fixed Point Theorem, (4.43) has at least one solution. Let
\[
|||f||| = \sup_{v \in H_0^1(\Lambda)} \frac{|(f, v)|}{|v|_1}.
\]
(4.43) has only one solution as long as \(|||f||| < \frac{1}{2} \mu^2\).

If (4.43) has a unique solution, then it is easy to solve it by various spectral approximations as presented in the previous parts. Here we are more interested in some problems with multi-solutions. In this case, it is better to consider (4.43) in an abstract framework. For convenience, we shall adopt a unified notation, i.e., \( \omega(x) \equiv 1 \) for the Legendre approximation or \( \omega(x) = (1 - x^2)^{-\frac{1}{2}} \) for the Chebyshev approximation. Let
\[
a_\omega(v, w) = \int_{\Lambda} \partial_x v(x) \partial_x (w(x) \omega(x)) \, dx
\]
and define the linear mapping \( A : (H_0^1(\omega(\Lambda)))' \rightarrow H_0^1(\omega(\Lambda)) \) by
\[
a_\omega(Av, w) = \langle v, w \rangle, \quad \forall \, w \in H_0^1(\omega(\Lambda)),
\] (4.44)

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( (H_0^1(\omega(\Lambda)))' \) and \( H_0^1(\omega(\Lambda)) \). It can be proved that \( A \) is a bounded mapping on \( H_0^1(\omega(\Lambda)) \), and a compact mapping from \( L^2(\omega(\Lambda)) \) into \( H_0^1(\omega(\Lambda)) \). Let \( \lambda = \frac{1}{\mu} \) and define the mapping \( G : \mathbb{R}^1 \times H_0^1(\omega(\Lambda)) \rightarrow L^2(\omega(\Lambda)) \) by
\[
G(\lambda, v) = \lambda (v \partial_x v - f).
\] (4.45)

Clearly \( G \) is infinitely differentiable. Let \( DG \) be the Fréchet derivative of \( G \). Then \( D^2G \) is bounded on any bounded subset of \( \mathbb{R}^1 \times H_0^1(\omega(\Lambda)) \). Moreover, since \( H_0^1(\omega(\Lambda)) \subset L^\infty(\Lambda) \), we get
\[
||G(\lambda, v)||_\omega \leq c|\lambda| (||v||_{L^\infty} ||v||_{L^1, \omega} + ||f||_\omega) \leq c|\lambda| (||v||_{L^1, \omega}^2 + ||f||_\omega).
\]

Finally, define the mapping \( F : \mathbb{R}^1 \times H_0^1(\omega(\Lambda)) \rightarrow H_0^1(\omega(\Lambda)) \), by
\[
F(\lambda, v) = v + AG(\lambda, v).
\]

Then problem (4.43) can be written equivalently as follows: find \( U \in H_0^1(\omega(\Lambda)) \) such that
\[
F(\lambda, U) = 0.
\] (4.46)
Let $\Lambda^*$ be any compact interval in $\mathbb{R}_+^\times = \{ \lambda \mid \lambda > 0 \}$. We can prove that there exists a branch of isolated solutions of (4.43), denoted by $(\lambda, U(\lambda))$. It means that there exists a positive constant $\delta$, such that
\[
||(I + ADG(\lambda, U(\lambda)))v||_{1,\omega} \geq \delta ||v||_{1,\omega}, \quad \forall \lambda \in \Lambda^*, \ v \in H_{0,\omega}^1(\Lambda).
\]

We begin to approximate the isolated solutions of (4.46). Let $P_N$ be the $L^2$-orthogonal projection and $P_N^{1,0}$ be the $H_{0,\omega}^1$-orthogonal projection, i.e.,
\[
a_\omega \left( v - P_N^{1,0}v, \phi \right) = 0, \quad \forall \phi \in \mathbb{R}^0_N.
\]
Set $A_N = P_N^{1,0}A$. Thanks to (4.44) and (4.47), for any $v \in \left( H_{0,\omega}^1(\Lambda) \right)'$,
\[
a_\omega(A_N v, \phi) = a_\omega(A v, \phi) = \langle v, \phi \rangle, \quad \forall \phi \in \mathbb{P}^0_N.
\]

Define
\[
F_N(\lambda, v) = v + A_N G(\lambda, v).
\]

Maday and Quarteroni (1981) proposed a spectral approximation to (4.46). It is to find $u_N \in \mathbb{P}^0_N$ such that
\[
F_N(\lambda, u_N) = 0.
\]
By using some results in Brezzi, Rappaz and Raviart (1980), the following result follows.

**Theorem 4.19.** Let $\{ (\lambda, U(\lambda)), \lambda \in \Lambda^* \}$ be a branch of isolated solutions of (4.46). There exists a neighborhood $\mathcal{B}$ of the origin in $H_{0,\omega}^1(\Lambda)$ and, for $N \geq N_0$, a unique $C^\infty$ mapping: $\lambda \in \Lambda^* \rightarrow u_N(\lambda) \in \mathbb{P}^0_N$ such that for any $\lambda \in \Lambda^*$, $u_N(\lambda)$ solves (4.48). Furthermore, if the mapping $\lambda \in \Lambda^* \rightarrow U(\lambda) \in H_{0,\omega}^1(\Lambda)$ is continuous for some $r > 1$, then for any $\lambda \in \Lambda^*$, there exists a constant $b$, depending on $||U(\lambda)||_{r,\omega}$ such that
\[
||U(\lambda) - u_N(\lambda)||_{1,\omega} + N||U(\lambda) - u_N(\lambda)||_{1,\omega} \leq bN^{1-r}.
\]
Lecture 5

Spectral Methods for Multi-dimensional and High Order Problems

In this lecture, we study the spectral methods in several spatial dimensions. We first list some results of orthogonal approximations in several dimensions. Next, we take the periodic problem of three-dimensional steady incompressible fluid flow and the stream function form of two-dimensional unsteady flow as two examples to describe the spectral methods for multi-dimensional problems and high order problems. The final part of this chapter is devoted to the spectral domain decomposition methods and the spectral multigrid methods. These techniques make the spectral methods more effective.

5.1. Orthogonal Approximations In Several Dimensions

We first consider the Fourier approximations. Let \( x = (x_1, \ldots, x_n) \) and \( \Omega = (0, 2\pi)^n \). Let \( l_p \) be integers and \( l = (l_1, l_2, \ldots, l_n) \). Set \( |l| = \max_{1 \leq q \leq n} |l_q| \) and \( l \cdot x = l_1 x_1 + \cdots + l_n x_n \). The set of functions \( e^{i_l \cdot x} \) is an orthogonal system in \( L^2(\Omega) \). The Fourier transformation of a function \( v \in L^2(\Omega) \) is as

\[
Sv(x) = \sum_{|l|=0}^{\infty} \hat{v}_l e^{i_l \cdot x}
\]

with the Fourier coefficients

\[
\hat{v}_l = \left( \frac{1}{2\pi} \right)^n \int_{\Omega} v(x)e^{-i_l \cdot x} \, dx, \quad |l| = 0, 1, \ldots
\]

Let \( N \) be any positive integer, and

\[
\hat{V}_N = \text{span} \{ e^{i_l \cdot x} \mid |l| \leq N \}.
\]

\( V_N \) denotes the subspace of all real-valued functions in \( \hat{V}_N \). In this subspace, some inverse inequalities are valid. Firstly, for any \( \phi \in \hat{V}_N \) and \( 1 \leq p \leq q \leq \infty \),

\[
\|\phi\|_{L^q} \leq c N^{\frac{1}{2}-\frac{1}{p}} \|\phi\|_{L^p}.
\]

Next, let \( m \) be a non-negative integer. For any \( \phi \in \hat{V}_N \) and \( |k| = m \),

\[
\|\partial_x^k \phi\|_{L^p} \leq (2N)^m \|\phi\|_{L^p},
\]

67
and for any $r \geq 0$, 
\[ \|\phi\|_r \leq c N^r \|\phi\|. \]

The $L^2$-orthogonal projection of a function $v \in L^2(\Omega)$ is 
\[ P_N v(x) = \sum_{|\ell| \leq N} \hat{v}_\ell e^{i\ell \cdot x}. \]

Let $H^r_p(\Omega)$ be the subspace of $H^r(\Omega)$ containing all functions with the period $2\pi$ for all variables. If $0 \leq \mu \leq r$, then for any $v \in H^r_p(\Omega)$, 
\[ \|v - P_N v\|_\mu \leq c N^{r-\mu} \|v\|_r. \]

$P_N v$ is also the $H^r$-orthogonal projection of $v \in H^r_p(\Omega)$. Now let $j = (j_1, j_2, \ldots, j_n)$, $x^{(j)} = (x_1^{(j_1)}, \ldots, x_n^{(j_n)})$ and 
\[ \Omega_N = \left\{ x^{(j)} \left| x_q^{(j_q)} = \frac{2\pi j_q}{2N+1}, 0 \leq j_q \leq 2N, 1 \leq q \leq n \right. \right\}. \]

The Fourier interpolant $I_N v(x)$ of a function $v \in C(\bar{\Omega})$ is such a trigonometric function in $\tilde{V}_N$ that 
\[ I_N v(x) = v(x), \quad x \in \Omega_N. \]

If $v \in H^r_p(\Omega), 0 \leq \mu \leq r$ and $r > \frac{n}{2}$, then 
\[ \|v - I_N v\|_\mu \leq c N^{r-\mu} \|v\|_r. \]

We next consider the Legendre approximations. Let $\Omega = (-1, 1)^n$. The Legendre polynomial of degree $l$ is 
\[ L_l(x) = \prod_{q=1}^n L_{l_q}(x_q). \]

The Legendre transformation of a function $v \in L^2(\Omega)$ is as 
\[ S v(x) = \sum_{|\ell| = 0}^\infty \hat{v}_\ell L_i(x) \]

with the Legendre coefficients 
\[ \hat{v}_l = \prod_{q=1}^n \left( l_q + \frac{1}{2} \right) \int_{\Omega} v(x) L_l(x) dx, \quad |l| = 0, 1, \ldots. \]

Let $P_N$ be the set of all polynomials of degree at most $N$ in all variables. Several inverse inequalities exist in $P_N$. For any $\phi \in P_N$ and $1 \leq p \leq q \leq \infty$, we have 
\[ \|\phi\|_{L_q} \leq c N^{2n\left(\frac{1}{p} - \frac{1}{q}\right)} \|\phi\|_{L_p}. \]

On the other hand, for any $\phi \in P_N, 2 \leq p \leq \infty$ and $|k| = m$, 
\[ \|\partial^k \phi\|_{L_p} \leq c N^{2m} \|\phi\|_{L_p}, \]
and for any \( r \geq 0, \)
\[
\|\phi\|_r \leq cN^{2r}\|\phi\|.
\]
The \( L^2 \)-orthogonal projection of a function \( v \in L^2(\Omega) \) is
\[
P_N v(x) = \sum_{|l|=0}^N \hat{v}_l L_l(x).
\]

If \( r \geq 0 \) and \( \mu \leq r \), then for any \( v \in H^r(\Omega), \)
\[
\|v - P_N v\| \mu \leq cN^{\sigma(\mu, r)}\|v\|_r,
\]
where \( \sigma(\mu, r) \) is given in (2.14). We introduce the inner product of \( H^m(\Omega) \), i.e.,
\[
(v, w)_m = \sum_{0 \leq |k| \leq m} \langle \partial_x^k v, \partial_x^k w \rangle.
\]
The \( H^m \)-orthogonal projection \( P_N^m : H^m(\Omega) \rightarrow \mathbb{P}_N \) is such a mapping that for any \( v \in H^m(\Omega), \)
\[
(v - P_N^m v, \phi)_m = 0, \quad \forall \phi \in \mathbb{P}_N.
\]
If \( v \in H^r(\Omega) \) and \( 0 \leq \mu \leq m \leq r \), then
\[
\|v - P_N^m v\|_\mu \leq cN^{r-\mu}\|v\|_r.
\]
We also need another kind of orthogonal projection in the numerical analysis of spectral methods. Let
\[
\mathbb{P}_N^{m,0} = \{ \phi \in \mathbb{P}_N \mid \partial_x^k \phi(x) = 0 \text{ on } \partial \Omega, |k| \leq m - 1 \},
\]
\[
a_m(v, w) = \sum_{|k|=m} \langle \partial_x^k v, \partial_x^k w \rangle.
\]
The \( H^m_0 \)-orthogonal projection \( P_N^{m,0} : H^m_0(\Omega) \rightarrow \mathbb{P}_N^{m,0} \) is such a mapping that for any \( v \in H^m_0(\Omega), \)
\[
a_m(v - P_N^{m,0} v, \phi) = 0, \quad \forall \phi \in \mathbb{P}_N^{m,0}.
\]
If \( v \in H^r(\Omega) \cap H^m_0(\Omega) \) and \( 0 \leq \mu \leq m \leq r \), then
\[
\|v - P_N^{m,0} v\|_\mu \leq cN^{m-r}\|v\|_r.
\]
Let \( x^{(j)} = (x^{(j)}_1, \ldots, x^{(j)}_n), 0 \leq j_q \leq N, x^{(j)}_q \) being the Legendre-Gauss type interpolation points. \( \Omega_N \) is the set of all \( x^{(j)} \). The Legendre interpolant \( I_N v(x) \in \mathbb{P}_N \) of a function \( v \in C(\Omega) \) is defined by
\[
I_N v \left( x^{(j)} \right) = v \left( x^{(j)} \right), \quad x \in \Omega_N.
\]
If \( v \in H^r(\Omega), r > \frac{n}{2} \) and \( 0 \leq \mu \leq r \), then
\[
\|v - I_N v\|_\mu \leq cN^{2\mu + \frac{\mu}{2} - r}\|v\|_r.
\]
In some special cases, we can improve the above result. For Legendre-Gauss interpolation and \( r > \frac{n}{2} \),
\[
\|v - I_N v\| \leq c N^{-r} \|v\|.
\]
For Legendre-Gauss-Lobatto interpolation and \( 0 \leq \mu \leq \min(1, 2r - n) \),
\[
\|v - I_N v\|_\mu \leq c N^{\mu - r} \|v\|.
\]

For the Chebyshev approximations in several dimensions, let \( \Omega = (-1, 1)^n \) and
\[
\omega(x) = \prod_{q=1}^{n} (1 - x_q^2)^{-\frac{1}{2}}.
\]
The Chebyshev polynomial of degree \( l \) is
\[
T_l(x) = \prod_{q=1}^{n} T_{l_q}(x_q).
\]
The Chebyshev transformation of a function \( v \in L^2_\omega(\Omega) \) is as
\[
Sv(x) = \sum_{|l|=0}^{\infty} \hat{v}_l T_l(x)
\]
with the Chebyshev coefficients
\[
\hat{v}_l = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \prod_{q=1}^{n} \frac{1}{c_{l_q}} \int_{\Omega} v(x) T_{l_q}(x) \omega(x) \, dx, \quad |l| = 0, 1, \ldots,
\]
where for \( 1 \leq q \leq n, c_{l_q} = 2 \) for \( l_q = 0 \) and \( c_{l_q} = 1 \) for \( l_q \geq 1 \). For any \( \phi \in \mathbb{P}_N \) and
\( 1 \leq p \leq q \leq \infty \),
\[
\|\phi\|_{L^2_\omega} \leq c N^{\frac{q-p}{2}} \|\phi\|_{L^p_\omega}
\]
Also for any \( \phi \in \mathbb{P}_N \), \( 2 \leq p \leq \infty \) and \( |k| = m \),
\[
\left\| \partial_x^k \phi \right\|_{L^p_\omega} \leq c N^{2m} \|\phi\|_{L^p_\omega}
\]
and for any \( \mu \geq 0 \),
\[
\|\phi\|_{H^\mu_\omega} \leq c N^{2\mu} \|\phi\|_{\omega}.
\]
The \( L^2_\omega \)-orthogonal projection of a function \( v \in L^2_\omega(\Omega) \) is
\[
P_N v(x) = \sum_{|l|=0}^{N} \hat{v}_l T_l(x).
\]
If \( r \geq 0 \) and \( \mu \leq r \), then for any \( v \in H^\mu_\omega(\Omega) \),
\[
\|v - P_N v\|_{H^\mu_\omega} \leq c N^{\sigma(\mu, r)} \|v\|_{H^\mu_\omega},
\]
\( \sigma(\mu, r) \) is given in (2.14). Furthermore, we define the inner product of \( H^m_\omega(\Omega) \) by
\[
(v, w)_{m, \omega} = \sum_{0 \leq |k| \leq m} \left( \partial_x^k v, \partial_x^k w \right)_{\omega}.
\]
The $H^m_\omega$-orthogonal projection $P^m_N : H^m_\omega(\Omega) \rightarrow \mathbb{P}_N$ is such a mapping that for any $v \in H^m_\omega(\Omega)$, 

\[(v - P^m_N v, \phi)_{n,\omega} = 0, \quad \forall \phi \in \mathbb{P}_N.\]

If $v \in H^r_\omega(\Omega)$ and $0 \leq \mu \leq 1 \leq r$, then 

\[\|v - P^1_N v\|_{\mu,\omega} \leq cN^{\mu - r}\|v\|_{r,\omega}.\]

Moreover, let 

\[\tilde{a}_{m,\omega}(v, w) = \sum_{|k| = m} (\partial^k_x v, \partial^k_x w)_{\omega}\]

and 

\[a_{m,\omega}(v, w) = \sum_{|k| = m} (\partial^k_x v, \partial^k_x (w \omega))_{\omega}.\]

In particular, $\tilde{a}_{\omega}(v, w) = \tilde{a}_{1,\omega}(v, \omega)$ and $a_{\omega}(v, w) = a_{1,\omega}(v, w)$. The $H^1_\omega$-orthogonal projection $\tilde{P}^1_N : H^1_\omega(\Omega) \rightarrow \mathbb{P}^0_N$ is such a mapping that for any $v \in H^1_\omega(\Omega)$, 

\[\tilde{a}_{\omega}(v - \tilde{P}^1_N v, \phi) = 0, \quad \forall \phi \in \mathbb{P}^0_N.\]

If $v \in H^r_\omega(\Omega) \cap H^1_\omega(\Omega)$ and $r \geq 1$, then 

\[\|v - \tilde{P}^1_N v\|_{1,\omega} \leq cN^{1 - r}\|v\|_{r,\omega}.\]

The other $H^1_\omega$-orthogonal projection $P^{1,0}_N : H^1_\omega(\Omega) \rightarrow \mathbb{P}^0_N$ is such a mapping that for any $v \in H^0_\omega(\Omega)$, 

\[a_{\omega}(v - P^{1,0}_N v, \phi) = 0, \quad \forall \phi \in \mathbb{P}^0_N.\]

If $v \in H^r_\omega(\Omega) \cap H^0_\omega(\Omega)$ and $0 \leq \mu \leq 1 \leq r$, then 

\[\|v - P^{1,0}_N v\|_{\mu,\omega} \leq cN^{\mu - r}\|v\|_{r,\omega}.\]

Let $x^{(j)} = (x^{(j)}_1, \ldots, x^{(j)}_n)$, $0 \leq j \leq N$, $x^{(j)}_q$ be the Chebyshev-Gauss type interpolation points, $\Omega_N$ is the set of all $x^{(j)}$. The Chebyshev interpolant $I_N v(x) \in \mathbb{P}_N$ of a function $v \in C(\Omega)$ is defined by 

\[I_N v(x^{(j)}) = v(x^{(j)}), \quad x^{(j)} \in \Omega_N.\]

If $v \in H^r_\omega(\Omega)$, $r > \frac{n}{2}$ and $0 \leq \mu \leq r$, then 

\[\|v - I_N v\|_{r,\omega} \leq cN^{2\mu - r}\|v\|_{r,\omega}.\]

We now turn to the orthogonal approximations on unbounded domains. First of all, let $\Omega = (0, \infty)^n$ and $\omega(x) = \exp\left(-\sum_{q=1}^{n} x_q\right)$. The Laguerre polynomial of degree $l$ is 

\[L_l(x) = \prod_{q=1}^{n} L_{l_q}(x_q).\]
The Laguerre transformation of a function \( v \in L^2_\omega(\Omega) \) is as
\[
Sv(x) = \sum_{|\ell|=0}^{\infty} \hat{\ell} \mathcal{L}_\ell(x)
\]
with the Laguerre coefficients
\[
\hat{\ell} = \int_\Omega v(x) \mathcal{L}_\ell(x) \omega(x) \, dx, \quad |\ell| = 0, 1, \ldots
\]
For any \( \phi \in \mathbb{P}_N \) and \( 1 \leq p \leq q \leq \infty \),
\[
\|\phi\|_{L^p_\omega} \leq cN^{\frac{q-p}{2}} \|\phi\|_{L^q_\omega}.
\]
Also for \( 1 \leq p \leq n \),
\[
\|\partial_p \phi\|_{\omega} \leq cN \|\phi\|_{\omega}.
\]
The \( L^2_\omega \)-orthogonal projection of a function \( v \in L^2_\omega(\Omega) \) is
\[
P_N v(x) = \sum_{|\ell|=0}^{N} \hat{\ell} \mathcal{L}_\ell(x).
\]
Let \( \alpha > 0 \) and
\[
H^r_\omega(\Omega, \alpha) = \left\{ v \in H^r_\omega(\Omega) \mid v \prod_{q=1}^{n} x_q^{\alpha} \in H^r_\omega(\Omega) \right\}
\]
with the norm
\[
\|v\|_{r, \omega, \alpha} = \left\| v \prod_{q=1}^{n} (1 + x_q)^{\alpha} \right\|_{r, \omega}.
\]
For any \( v \in H^r_\omega(\Omega, \alpha) \) and \( 0 \leq \mu \leq r \),
\[
\|v - P_N v\|_{\mu, \omega} \leq cN^{\mu - \frac{r}{2}} \|v\|_{r, \omega, \alpha}
\]
where \( \alpha \) is the largest integer such that \( \alpha < r + 1 \).

Finally let \( \Omega = (-\infty, \infty)^n \) and \( \omega(x) = \exp \left( -\sum_{q=1}^{n} x_q^2 \right) \). The Hermite polynomial of degree \( l \) is
\[
H_l(x) = \prod_{q=1}^{n} H_{i_q}(x_q).
\]
The Hermite transformation of a function \( v \in L^2_\omega(\Omega) \) is as
\[
v(x) = \sum_{|\ell|=0}^{\infty} \hat{\ell} H_l(x)
\]
with the Hermite coefficients
\[
\hat{\ell} = (\pi)^{-\frac{n}{2}} \left( \prod_{q=1}^{n} 2^{l_q} (l_q)! \right)^{-\frac{1}{2}} \int_\Omega v(x) H_l(x) \omega(x) \, dx, \quad |\ell| = 0, 1, \ldots
\]
For any $\phi \in \mathcal{P}_N$ and $1 \leq p \leq q \leq \infty$,
\[
\|\phi\|_{L^q_p} \leq cN^{\frac{2p}{q} - 1 + \frac{1}{q}} \|\phi\|_{E_N^q}.
\]
Also for $1 \leq p \leq n$,
\[
\|\partial_p \phi\|_\omega \leq c\sqrt{N} \|\phi\|_\omega.
\]
The $L^2_\omega$-orthogonal projection of $v \in L^2_\omega(\Omega)$ is
\[
P_N v(x) = \sum_{|\ell| = 0}^N \hat{v}_\ell H_\ell(x).
\]
For any $v \in H^r_\omega(\Omega)$ and $0 \leq \mu \leq r$, we have
\[
\|v - P_N v\|_{\mu, \omega} \leq cN^{\frac{r - \mu}{2}} \|v\|_{r, \omega}.
\]
Some of the above results can be found in Canuto and Quarteroni (1982), and Bernardi and Maday (1992, 1997). The rest can be found in Guo (1998).

The filtering and the techniques for recovering the spectral accuracy in Section 2.7 can be generalized to the orthogonal approximations in several dimensions. For example, let $\Omega = (0, 2\pi)^n$ and $|l| = \left(\sum_{q=1}^n r_q^2\right)^{\frac{1}{2}}$. The filtering $R_N(\alpha, \beta)$ can be defined as
\[
R_N(\alpha, \beta) \phi(x) = \sum_{|\ell| \leq N} \prod_{q=1}^n \left(1 - \frac{|l|_q^2}{N^\alpha}\right)^\beta \hat{\phi}_\ell e^{i\ell \cdot x}, \quad \alpha, \beta \geq 1, \phi \in V_N.
\]
For any $\phi \in \tilde{V}_N$ and $0 \leq r - \mu \leq \alpha$,
\[
\|R_N(\alpha, \beta) \phi - \phi\|_\mu \leq c\beta N^{\alpha - r} \|\phi\|_r.
\]
If in addition $\mu \geq 0$, then
\[
\|R_N(\alpha, \beta) \phi - \phi\|_\mu \leq c\beta N^{\alpha - r} \|\phi\|_r.
\]
For the sake of simplicity, we still use the notations $L^\infty(\Omega), L^q_\omega(\Omega), W^{\alpha, r}_\omega(\Omega)$ for the spaces of vector functions in the following discussions.

5.2. Spectral Methods For Multi-Dimensional Nonlinear Systems

The aim of this part is to present the spectral methods for multi-dimensional nonlinear systems. In particular, we are interested in spectral approximations of isolated solutions. For simplicity, we focus on the three-dimensional periodic problem of steady Navier-Stokes equation.

Navier-Stokes equation is the fundamental equation describing incompressible fluid flows. Let $\Omega = (0, 2\pi)^3$ and $L^2_\omega(\Omega)$ be the subspace of $L^2(\Omega)$, containing all functions with the zero average over $\Omega$. Denote by $U(x), P(x)$ and $\mu > 0$ the speed vector, the pressure and the kinetic viscosity respectively. The components
of $U(x)$ are $U^{(j)}(x), 1 \leq j \leq 3$. Moreover $f(x)$ stands for the body force, and $f(x) = (f^{(1)}(x), f^{(2)}(x), f^{(3)}(x))$. The steady Navier-Stokes equation is of the form

$$\begin{cases} (U \cdot \nabla)U - \mu \Delta U + \nabla p = f, & x \in \Omega, \\ \nabla \cdot U = 0, & x \in \Omega. \end{cases} \quad (5.1)$$

Assume that all functions have the period $2\pi$ for all variables. For fixing the values of $P$, we require that $P \in L_0^2(\Omega)$. Let $V = \{ v \mid v \in D_p(\Omega), \nabla \cdot v = 0 \}$ and $\bar{V}$ be the closure of $V$ in $H^1(\Omega)$. Set

$$a(v, w) = \int_{\Omega} \nabla v(x) \cdot \nabla w(x) \, dx,$$

$$b(v, w, z) = \int_{\Omega} v(x) \cdot (w(x) \cdot \nabla)z(x) \, dx.$$

A weak solution of (5.1) is defined as a pair $(U, P) \in \bar{V} \times L_0^2(\Omega)$, satisfying

$$\mu a(U, v) - \tilde{b}(U, U, v) = (f, v), \quad \forall v \in H^1_0(\Omega). \quad (5.2)$$

If $(f, v)$ is a linear continuous functional in $\bar{V}$, then (5.2) has at least one weak solution.

If (5.2) has a unique solution, then it is not difficult to approximate it directly by the Fourier spectral method or the Fourier pseudospectral method. But in those cases, the trial functions $\phi_l$ have to satisfy the incompressibility condition, i.e., $\nabla \cdot \phi_l = 0$. It is not convenient in actual computations. We can also approximate (5.1) with some trial functions whose divergences do not vanish. However we need to evaluate $U$ coupled with $P$. In order to remove this trouble, we try to eliminate the pressure $P$ in the first formula of (5.1). To do this, let

$$\phi_l(x) = \left( \frac{1}{\sqrt{2\pi}} \right)^3 e^{i l \cdot x}$$

and for any $v \in L^2(\Omega)$,

$$\hat{v}_l = (v, \phi_l), \quad |l| = 0, 1, \ldots,$$

Then

$$\int_{\Omega} ((U(x) \cdot \nabla)U(x) - f(x)) \cdot \nabla \phi_l(x) \, dx = \int_{\Omega} P(x) \Delta \phi_l(x) \, dx = -|l|^2 \int_{\Omega} P(x) \phi_l(x) \, dx \quad (5.3)$$

where $|l|^2 = \sum_{q=1}^n l_q^2$. By (5.3) and $P \in L_0^2(\Omega)$,

$$\begin{cases} (P, \phi_0) = 0, \\ (P, \phi_l) = -\frac{1}{|l|^2} \int_{\Omega} ((U(x) \cdot \nabla)U(x) - f(x)) \cdot \nabla \phi_l(x) \, dx, & \text{for } l \neq 0. \end{cases} \quad (5.4)$$

Set

$$b(v, w, z) = \int_{\Omega} [(w(x) \cdot \nabla)v(x)] \cdot z(x) \, dx$$
\[- \sum_{t \neq 0} \frac{l \cdot \bar{z}_t}{|l|^2} \int_{\Omega} (l \cdot (w(x) \cdot \nabla)v(x))\bar{\phi}_t(x) \, dx, \]

\[\mathcal{F}(z) = \int_{\Omega} f(x) \cdot z(x) \, dx - \sum_{t \neq 0} \frac{l \cdot \bar{z}_t}{|l|^2} (l \cdot f, \phi_t).\]

Then we can derive a new representation of (5.2). It is to find \(U \in H^1_p(\Omega)\) such that

\[\mu a(U, v) + b(U, U, v) = \mathcal{F}(v), \quad \forall \, v \in H^1_p(\Omega),\]

while \(P\) is determined by (5.4).

We can rewrite (5.2) in another way. To do this, let

\[b^0(v, w, z) = i \sum_{t \neq 0} \int_{\Omega} (w(x) \cdot l)(v(x) \cdot \bar{z}_t) \bar{\phi}_t(x) \, dx\]

\[-i \sum_{t \neq 0} \frac{l \cdot \bar{z}_t}{|l|^2} \int_{\Omega} (w(x) \cdot l)(v(x) \cdot l)\bar{\phi}_t(x) \, dx,\]

\[b^1(v, w, z) = -\int_{\Omega} (\nabla \cdot w(x))(v(x) \cdot \bar{z}(x)) \, dx + \sum_{t \neq 0} \frac{l \cdot \bar{z}_t}{|l|^2} (\nabla \cdot w, (\bar{v}(x) \cdot l) \phi_t).\]

Then

\[b(v, w, z) = b^0(v, w, z) + b^1(v, w, z), \quad \forall \, v, w \in H^1_p(\Omega).\]

Moreover, if in addition \(w \in V\), then \(b^1(v, w, z) = 0\). Thus (5.2) can be presented as follows. It is to find \(U \in H^1_p(\Omega)\), such that

\[\mu a(U, v) + b^0(U, U, v) = \mathcal{F}(v), \quad \forall \, v \in H^1_p(\Omega).\]

**Theorem 5.1.** The representation (5.1) is equivalent to (5.5) with (5.4), or to (5.6) with (5.4).

We are going to construct the Fourier-spectral scheme. For this purpose, we put (5.5) in an abstract framework. Let \(\lambda = \frac{1}{p}\) and \(E(\Omega) = \left(H^\frac{2}{p}(\Omega)^{'}, \right)\). Since \(H^\frac{2}{p}(\Omega) \subset H^\frac{2}{p}(\Omega)\), we have that \(E(\Omega) \subset H^{-\frac{2}{p}}(\Omega)\) and also \(E(\Omega) \subset (H^1_p(\Omega))^{'}\). We define an inner product on \(H^1_p(\Omega)\) by

\[(v, w)_1 = a(v, w) + (v, w), \quad \forall \, v, w \in H^1_p(\Omega),\]

and a linear mapping \(A : (H^1_p(\Omega))^{'} \to H^1_p(\Omega)\) by

\[A(v, w)_1 = (v, w), \quad \forall \, w \in H^1_p(\Omega).\]

Moreover by the theory of linear elliptic equations, \(A\) is a continuous mapping from \(H^r(\Omega)\) with \(r \geq -1\) into \(H^{r+2}(\Omega) \cap H^1_p(\Omega)\). Since \(E(\Omega) \subset H^{-\frac{2}{p}}(\Omega)\), \(A\) is continuous from \(E(\Omega)\) into \(H^\frac{2}{p}(\Omega) \cap H^1_p(\Omega)\). But the latter space is compactly contained in \(H^1_p(\Omega)\), whence \(A\) is a compact mapping from \(E(\Omega)\) into \(H^1_p(\Omega)\). Define \(G : \mathbb{R}^1 \times H^1_p(\Omega) \to E(\Omega)\) by

\[G(\lambda, v) = \lambda (b(v, v, w) - \mathcal{F}(w)) - (v, w).\]
We know that for any $\lambda \in \mathbb{R}^1$ and $v \in H^1_p(\Omega),$

$$||G(\lambda, v)||_E \leq c|\lambda|\left(||v||^2 + ||f||\right).$$

Moreover $G$ is a $C^\infty$ mapping. Let $D$ be the Fréchet derivative of $G$. Then for any positive integer $m$, $D^m G$ is bounded over any bounded subset of $\mathbb{R}^1 \times H^1_p(\Omega)$. According to (5.7) and (5.8), (5.5) is equivalent to

$$(U, v)_1 + \langle G(\lambda, U), v \rangle = 0, \quad \forall \ v \in H^1_p(\Omega).$$

Setting $F : \mathbb{R}^1 \times H^1_p(\Omega) \rightarrow H^1_p(\Omega)$,

$$F(\lambda, v) = v + A G(\lambda, v).$$

By (5.7) and (5.8), we get an equivalent form of (5.5). It is to find $U \in H^1_p(\Omega)$ such that

$$F(\lambda, U) = 0. \quad (5.9)$$

Let $\Lambda^*$ be a compact interval in $\mathbb{R}^1$, and assume that $(\lambda, U(\lambda))$ is a branch of isolated solutions of (5.9). So there exists $\delta > 0$ such that for all $\lambda \in \Lambda^*$ and $v \in H^1_p(\Omega),$

$$||I + ADG(\lambda, U(\lambda))v||_1 \geq \delta ||v||_1.$$ 

Now let $V_N$ be the same as in the previous part, and $P_N$ be the $L^2$-orthogonal projection upon $V_N$. $P_N$ is a continuous mapping in $H^1_p(\Omega)$ with $r \geq 0$. Let $A_N = P_N A$. Clearly $A_N \in \mathcal{L}\left((H^1_p(\Omega))^{'}, V_N\right)$ and

$$(A_N v, \phi)_1 = (v, \phi), \quad \forall \ \phi \in V_N. \quad (5.10)$$

Setting $F_N : \mathbb{R}^1 \times V_N \rightarrow V_N$,

$$F_N(\lambda, v) = v + A_N G(\lambda, v).$$

Maday and Quarteroni (1982b) provided a Fourier-spectral approximation to (5.9). It is to find $u_N \in V_N$ such that

$$F_N(\lambda, u_N) = 0. \quad (5.11)$$

It is equivalent to finding $u_N \in V_N$ such that

$$a(u_N, \phi) + \lambda b(u_N, u_N, \phi) = \lambda f(\phi), \quad \forall \ \phi \in V_N.$$ 

It can be shown that $\nabla \cdot u_N = 0$. We have the following result.

**Theorem 5.2.** Let $\{\lambda, U(\lambda)\}, \lambda \in \Lambda^*$ be a branch of isolated solutions of (5.9). There exists a neighborhood $\ominus$ of the origin in $H^1_p(\Omega)$, and for $N$ sufficiently large, a unique $C^\infty$ mapping $\lambda \in \Lambda^* \rightarrow u_N(\lambda) \in V_N$ such that for any $\lambda \in \Lambda^*$, $F_N(\lambda, u_N(\lambda)) = 0$ and $U(\lambda) - u_N(\lambda) \in \ominus$. Moreover, if the mapping $\lambda \in \Lambda^* \rightarrow U(\lambda) \in H^1_p(\Omega)$ is continuous for some $r \geq 1$, then for any $\lambda$, there is a positive constant $b_1$ depending on $||U(\lambda)||_r$ such that

$$||U(\lambda) - u_N(\lambda)||_1 + N ||U(\lambda) - u_N(\lambda)|| \leq b_1 N^{1-r}.$$
The approximate pressure \( p_N(\lambda) = p_N(x, \lambda) \) could be determined by

\[
p_N(\lambda) = \sum_{|l| \leq N} \hat{p}_{N,l}(\lambda) \phi_l(\lambda)
\]

with

\[
\hat{p}_{N,0}(\lambda) = 0, \quad \hat{p}_{N,l}(\lambda) = \frac{i}{|l|^2} \left( l \cdot \left( \frac{1}{N} \sum_{x \in \Omega} \phi_l(x) u_N(x, \lambda) - f \right), \phi_l \right), \quad l \neq 0.
\]

**Theorem 5.3.** Assume that the hypotheses of Theorem 5.2 hold, \( P(\lambda) \in H^1_0(\Omega) \cap L_1^2(\Omega) \), and \( r \geq 1 \). Then there is a positive constant \( b_2 \) depending on \( \|U(\lambda)\| \), such that

\[
\|P(\lambda) - p_N(\lambda)\| + N^{-1}\|P(\lambda) - p_N(\lambda)\|_1 \leq b_2 N^{-r}.
\]

We now consider the Fourier-pseudospectral approximation. Define the mapping \( \tilde{G}: \mathbb{R}^1 \times H^1_0(\Omega) \rightarrow E(\Omega) \) by

\[
\langle \tilde{G}(\lambda, v), w \rangle = \lambda \left( b^0(v, v, w) - \mathcal{F}(w) \right) - (v, w).
\]

The mapping defined by (5.12) has the properties of \( G \) defined by (5.8). In particular, (5.6) can be equivalently written in the form (5.9) where \( \tilde{G} \) is given by (5.12).

Now, let \( \Omega_N \) be the set of all Fourier interpolation points. For any \( v, w \in C(\Omega) \), the discrete inner product is given by

\[
(v, w)_N = \left( \frac{2\pi}{2N+1} \right)^3 \sum_{x \in \Omega_N} v(x) \bar{w}(x).
\]

Denote by \( I_N v \) and \( \bar{w} \) the Fourier interpolant of \( v \in C(\Omega) \) and the Fourier coefficients. Assume that \( f \in C(\Omega) \). Define \( b^0_N(v, w, z) : V_N \rightarrow C(\Omega) \) and \( \mathcal{F}_N(w) : V_N \rightarrow C(\Omega) \) by

\[
b^0_N(v, w, z) = i \sum_{|l| \leq N, l \neq 0} \left( (w \cdot l) (v \cdot \bar{z}_l), \phi_l \right)_N - \frac{1}{|l|^2} \left( (w \cdot l)(v \cdot l), \phi_l \right)_N,
\]

\[
\mathcal{F}_N(w) = (f, w)_N - \sum_{|l| \leq N, l \neq 0} \frac{1}{|l|^2} \left( l \cdot \bar{w}_l \right) (l \cdot f, \phi_l)_N.
\]

Further, define \( G_N : \mathbb{R}^1 \times V_N \rightarrow V_N \) as

\[
\langle G_N(\lambda, v), w \rangle = \lambda \left( b^0_N(v, v, w) - \mathcal{F}(w) \right) - (v, w)_N,
\]

and let

\[
\langle G_N(\lambda, v), w \rangle = \langle \tilde{G}_N(\lambda, v), P_N w \rangle, \quad \forall \lambda \in \mathbb{R}^1, \ v \in V_N, \ w \in H^1_0(\Omega).
\]

Let \( A_N \) be the same as in (5.10) and set \( F_N : \mathbb{R}^1 \times V_N \rightarrow V_N \),

\[
F_N(\lambda, v) = v + A_N G_N(\lambda, v).
\]
A Fourier-pseudospectral approximation of (5.6) is to find \( u_N \in V_N \) such that

\[
F_N(\lambda, u_N) = 0.
\]  

(5.13)

It is equivalent to finding \( u_N \in V_N \) such that

\[
a(u_N, \phi) + \lambda b_N(u_N, u_N, \phi) = \lambda F_N(\phi), \quad \forall \phi \in V_N.
\]

It can be checked that \( \nabla \cdot u_N = 0 \). We have the following result.

**Theorem 5.4.** Let \( \{(\lambda, U(\lambda)), \lambda \in \Lambda^*\} \) be a branch of isolated solutions of (5.6), and \( r_1 > \frac{3}{2}, r_2 > 2 \). If \( f \in H^r_1(\Omega) \) and the mapping \( \lambda \in \Lambda^* \rightarrow U(\lambda) \in H^r_1(\Omega) \) is continuous, then for \( N \geq N_0 \) large enough, there exists a positive constant \( b_3 \) depending on \( \|U(\lambda)\|_{r_2} \) and \( \|f\|_{r_1} \), and a unique \( C^\infty \) mapping \( \lambda \in \Lambda^* \rightarrow u_N(\lambda) \in V_N \) such that for any \( \lambda \in \Lambda^* \), \( F_N(\lambda, u_N(\lambda)) = 0 \) and

\[
\|U(\lambda) - u_N(\lambda)\|_1 \leq b_3 \left( N^{1-r_2} + N^{-r_1} \right).
\]

In order to obtain an improved \( L^2 \)-error estimate, let \( M > N \) be a suitably chosen integer. The corresponding set of Fourier interpolation points, the discrete inner product and the Fourier interpolation are denoted by \( \Omega_M, (\cdot, \cdot)_M \) and \( I_M \). If \( M > N^{\frac{1}{r-2}} \), then

\[
\|f - I_M f\| \leq c M^{1-r} \|f\|_{r-1} \leq c N^{-r} \|f\|_{r-1}.
\]

Define the mapping \( G_N^*(\lambda, v) \) as

\[
G_N^*(\lambda, v) = G_N(\lambda, v) + \lambda (\mathcal{F}_N(v) - \mathcal{F}_N(\phi))
\]

where \( \mathcal{F}_N : V_N \rightarrow C(\bar{\Omega}) \) is given by

\[
\mathcal{F}_N(v) = (f, v)_M - \sum_{|l| \leq N, i \neq 0} \frac{1}{|l|^2} (l \cdot \overline{v}_i)(l \cdot f, \phi_i)_M.
\]

Let \( F_N(\lambda, v) = v + A_N G_N^*(\lambda, v) \). A new Fourier pseudospectral approximation is to find \( u^*_N \in V_N \) such that

\[
F_N^*(\lambda, u^*_N(\lambda)) = 0.
\]  

(5.14)

We have the following result.

**Theorem 5.5.** Let \( \{(\lambda, U(\lambda)), \lambda \in \Lambda^*\} \) be a branch of isolated solutions of (5.6) and \( u^*_N \) be the solution of (5.14). If \( f \in H^{r-1} \) with \( r > 2 \) and the mapping \( \lambda \in \Lambda^* \rightarrow U(\lambda) \in H^r_1(\Omega) \cap H^r_\theta(\Omega) \) is continuous, then there exists a positive constant \( b_4 \) depending on \( \|U(\lambda)\|_r \) and \( \|f\|_{r-1} \) such that for any \( \lambda \in \Lambda^* \),

\[
\|U(\lambda) - u^*_N(\lambda)\| \leq b_4 N^{-r}.
\]

The approximate pressure \( p_N^*(\lambda) \) is given by

\[
p_N^*(\lambda) = \sum_{|i| \leq N} \hat{p}_{N,i}^*(\lambda) \phi_i(x).
\]
with \( \hat{p}_{N,0}^*(\lambda) = 0 \) and for \( l \neq 0 \),
\[
\hat{p}_{N,l}^*(\lambda) = \frac{-1}{|l|^2} ((u_N^*(\lambda) \cdot l) (u_N^*(\lambda) \cdot l) + i f \cdot l, \phi_l)_M.
\]
If \( M > N^{\frac{r+1}{r}} \), then
\[
\| P(\lambda) - p_N^*(\lambda) \| + N^{-1} \| P(\lambda) - p_N^*(\lambda) \| \leq b_6 N^{-r}
\]
where \( b_6 \) is a positive constant depending on \( \| U(\lambda) \|_r \) and \( \| f \|_{r-1} \).

For the Fourier approximations of unsteady Navier-Stokes equation, we refer to the early work in Orszag (1971a, 1971b), Hald (1981), Kuo (1983), Guo (1985), Guo and Ma (1987), and Ma and Guo (1987) provided various schemes with numerical analysis for computational fluid dynamics.

5.3. Spectral Methods For Nonlinear High Order Equations

Many nonlinear partial differential equations of high order occur in science and engineering, such as the Korteweg-de Vries equation possessing solitons, the Landau-Ginzburg equation in the dynamics of solid-state phase transitions in shape memory alloys, etc. In this part, we consider the spectral methods for such kind of equations. We shall take the initial-boundary value problem of two-dimensional Navier-Stokes equation in stream function representation as an example to describe the Legendre spectral method for solving initial-boundary value problems of nonlinear revolutionary equations of fourth order.

Let \( \Omega = (-1,1)^2 \) and \( \partial \Omega \) be a non-slip wall. The initial-boundary value problem of two-dimensional Navier-Stokes equation is of the form
\[
\begin{align*}
\partial_t U + (U \cdot \nabla) U - \mu \Delta U + \nabla P &= f, & x \in \Omega, 0 < t \leq T, \\
\nabla \cdot U &= 0, & x \in \Omega, 0 \leq t \leq T, \\
U &= 0, & x \in \partial \Omega, 0 < t \leq T, \\
U(x,0) &= U_0(x), & x \in \Omega.
\end{align*}
\] (5.15)

Let \( \mathcal{V} = \{ v \mid v \in \mathcal{D}(\Omega), \nabla \cdot v = 0 \} \). The closures of \( \mathcal{V} \) in \( L^2(\Omega) \) and \( H^1(\Omega) \) are denoted by \( \mathcal{H} \) and \( \mathcal{V} \) respectively. Let
\[
a(v, w) = \int_{\Omega} \nabla v(x) \cdot \nabla w(x) \, dx,
\]
\[
b(v, w, z) = \int_{\Omega} (w(x) \cdot \nabla v(x)) \cdot z(x) \, dx.
\]

The weak form of (5.15) is as follows
\[
\begin{align*}
\{ (\partial_t U, v) + b(U, U, v) + \mu a(U, v) = (f, v), & \forall v \in \mathcal{V}, 0 < t \leq T, \\
U(0) = U_0, & x \in \Omega.
\end{align*}
\] (5.16)

If \( U_0 \in \mathcal{H} \) and \( f \in L^2(0,T;\mathcal{V}) \), then (5.16) has a unique solution in \( L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) \). The solution of (5.16) possesses the conservation
\[
\| U(t) \|^2 + 2\mu \int_0^t \| U(s) \|^2 \, ds = \| U_0 \|^2 + 2 \int_0^t (f(s), U(s)) \, ds.
\] (5.17)
For numerical simulations, we can approximate (5.16) directly. But in this case, we
should construct a trial function space whose elements satisfy the incompressibility. It
is not convenient in actual computations. To avoid it, several alternative expressions
are concerned. Firstly, we can take the divergences of both sides of (5.15), and get
\[
\begin{cases}
\partial_t U + (U \cdot \nabla) U - \mu \Delta U + \nabla P = f, & x \in \Omega, \ 0 < t \leq T, \\
\Delta P + \Phi(U) = \nabla \cdot f, & x \in \Omega, \ 0 < t \leq T
\end{cases}
\] (5.18)
where
\[
\Phi(U) = 2 \left( \partial_2 U^{(1)} \partial_1 U^{(2)} - \partial_1 U^{(1)} \partial_2 U^{(2)} \right).
\]
Conversely if \( U \) and \( P \) satisfy (5.18) and \( \nabla \cdot U(0) = 0 \), then for all \( t \geq 0, \nabla \cdot U = 0 \).
Thus (5.18) is equivalent to (5.15) provided that \( U \) and \( P \) are suitably smooth. The
second formula of (5.18) is a Poisson equation for \( P \) at each time \( t \). If we use it to
evaluate the pressure, then we need certain boundary conditions for the pressure. We
have that
\[
\partial_n P(x,t) = \nu \cdot (\mu \Delta U(x,t) + f(x,t)), \quad x \in \partial \Omega, \ 0 < t \leq T
\]
where \( \partial_n P \) is the outward normal derivative of \( P \). A simplified condition is
\[
\partial_n P(x,t) = N(x,t), \quad x \in \partial \Omega, \ 0 < t \leq T.
\]
But the first condition is too complicated in actual calculations, while the simplified
one is not physical.

The second way is to introduce the vorticity \( H(x,t) \), the stream function \( \Psi(x,t) \) and
\[
J(v,w) = \partial_2 w \partial_1 v - \partial_1 w \partial_2 v, \quad \forall \ v, w \in H^1(\Omega).
\]
Then (5.15) is equivalent to
\[
\begin{cases}
\partial_t H + J(H, \Psi) - \mu \Delta H = \nabla \times f, & x \in \Omega, \ 0 < t \leq T, \\
-\Delta \Psi = H, & x \in \Omega, \ 0 \leq t \leq T, \\
H(x,0) = H_0(x), & x \in \bar{\Omega}.
\end{cases}
\] (5.19)
Since the incompressibility is already included, we do not need to construct the trial
function space with free-divergence. Clearly \( \Psi(x,t) = 0 \) on the boundary. But it is
not easy to deal with the boundary values of the vorticity. In some literature, it is
assumed approximately that \( H(x,t) = 0 \) on \( \partial \Omega \). But it is not physical.

The third representation of Navier-Stokes equation is the stream function form. Let
\[
G(v,w) = \partial_2 v \partial_1 (\Delta w) - \partial_1 v \partial_2 (\Delta w).
\]
Then the stream function form is as follows
\[
\begin{cases}
\partial_t \Psi + G(\Psi, \Psi) - \mu \Delta^2 \Psi = g, & x \in \Omega, \ 0 < t \leq T, \\
\Psi + \partial_n \Psi = 0, & x \in \Omega, \ 0 < t \leq T, \\
\Psi(x,0) = \Psi_0(x), & x \in \bar{\Omega}
\end{cases}
\] (5.20)
where \( g = -\nabla \times f \). The main merits of this expression are remedying the troubles
mentioned in the above and keeping the physical boundary values naturally.
We shall derive the weak form of (5.20). For any \( v, z \in W^{1,4}(\Omega) \) and \( w \in H^2(\Omega) \), let
\[
J(v, w, z) = (\Delta w, \partial_2 v \partial_1 z - \partial_1 v \partial_2 z).
\]
Obviously
\[
J(v, w, z) + J(z, w, v) = 0, \quad \text{Equation (5.21)}
\]
\[
J(v, w, z) = -(G(v, w), z). \quad \text{Equation (5.22)}
\]

The weak solution of (5.20) is a function \( \Psi \in L^2(0, T; H^2_0) \cap L^\infty(0, T; H^1) \) such that
\[
\begin{aligned}
&\{ \left. \begin{array}{l}
(\partial_t \nabla \Psi(t), \nabla v) + J(\Psi(t), \Psi(t), v) \\
\quad + \mu(\nabla \Psi(t), \nabla v) + (g(t), v)_{L^1(H^{-1}(\Omega), H^1_0(\Omega))} = 0, \quad \forall \, v \in H^2_0(\Omega), \quad 0 < t \leq T, \\
\Psi(x, 0) = \Psi_0(x),
\end{array} \right. \\
&\quad \forall \, x \in \Omega.
\end{aligned}
\]
\[
\text{Equation (5.23)}
\]
It is shown in Guo, He and Mao (1997) that if \( \Psi_0 \in H^2_0(\Omega) \) and \( g \in L^2(0, T; H^{-2}) \), then (5.23) has a unique solution. The solution satisfies the conservation
\[
||\nabla \Psi(t)||^2 + 2\mu \int_0^t ||\Delta \Psi(s)||^2 \, ds + s \int_0^t \langle g(s), \Psi(s) \rangle_{L^1(\Omega), H^{-1}(\Omega)} \, ds = ||\nabla \Psi_0||^2.
\]
\[
\text{Equation (5.24)}
\]

We can adopt Legendre approximations or Chebyshev approximations to solve (5.23) numerically. For the sake of simplicity, we focus on the Legendre spectral method. Let \( P^2_N = P_N \cap H^2_0(\Omega) \) and
\[
\begin{aligned}
&(v, w)_1 = (\nabla v, \nabla w), \quad \forall \, v, w \in H^1_0(\Omega), \\
&(v, w)_2 = (\Delta v, \Delta w), \quad \forall \, v, w \in H^2_0(\Omega).
\end{aligned}
\]
The orthogonal projections \( \tilde{P}^m_0 : H^m_0(\Omega) \rightarrow P^2_N, m = 1, 2 \), are two such mappings that for any \( v \in H^m_0(\Omega) \),
\[
\left( v - \tilde{P}^{m,0}_N v, \phi \right)_m = 0, \quad \forall \, \phi \in P^{2,0}_N, \, m = 1, 2.
\]

It is not difficult to show that for any \( v \in H^1_0(\Omega) \), \( ||v - \tilde{P}^{1,0}_N v||_1 \rightarrow 0 \) as \( N \rightarrow \infty \). Now let \( \psi_N \) be the approximation to \( \Psi \). A Legendre spectral scheme for (5.23) is to find \( \psi_N(\xi) \in P^{2,0}_N \) for all \( t \in \tilde{R}_r(T) \) such that
\[
\begin{aligned}
&\{ \left. \begin{array}{l}
(D, \nabla \psi_N(t), \nabla \phi) + J(\psi_N(t), \psi_N(t), \psi_N(t), \phi) \\
\quad + \mu(\nabla \psi_N(t), \nabla \phi) + (g(t), \phi)_{L^1(H^{-1}(\Omega), H^1_0(\Omega))} = 0, \quad \forall \, \phi \in P^{2,0}_N, \quad t \in \tilde{R}_r(T), \\
\psi(0) = \psi_{N,0}(x),
\end{array} \right. \\
&\quad \forall \, x \in \Omega.
\end{aligned}
\]
\[
\text{Equation (5.25)}
\]
where \( \sigma, \delta \) are parameters, \( 0 \leq \sigma, \delta \leq 1 \) and
\[
||\Psi_0 - \psi_{N,0}||_1 \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.
\]
\[
\text{Equation (5.26)}
\]
For \( \Psi_0 \in H^2_0(\Omega) \), we take \( \psi_{N,0} = \tilde{P}^{1,0}_N \Psi_0 \). While for \( \Psi_0 \in H^2_0(\Omega) \), we take \( \psi_{N,0} = \tilde{P}^{2,0}_N \Psi_0 \). Both of them satisfy (5.26). Thus the approximations on the initial level
are well defined. Also, we can prove that (5.25) has a unique solution as long as \( \Psi_0 \in H^0_0(\Omega) \) and \( g \in L^2(0, T; L^2(\Omega)) \). Especially, if \( \sigma = \delta = \frac{1}{2}, \) then

\[
||\nabla \psi_N(t)||^2 + \frac{1}{2} \mu^\tau \sum_{s \in \mathbb{R}_+ \{t - \tau\}} ||\Delta \psi_N(s) + \Delta \psi_N(s + \tau)||^2 + \tau \sum_{s \in \mathbb{R}_+ \{t - \tau\}} (g(s), \psi_N(s) + \psi_N(s + \tau)) \, ds = ||\nabla \psi_N_0||^2
\]

which is a reasonable simulation of (5.24).

In practical computations, a good choice of basis functions is essentially important. Shen (1994) proposed a nice set of basis functions. Let \( L_m(y) \) be the Legendre polynomial of degree \( m \). Set

\[
\chi_m(y) = d_m \left( L_m(y) - \frac{2(2m + 5)}{2m + 7} L_{m+2}(y) + \frac{2m + 3}{2m + 7} L_{m+4}(y) \right),
\]

\[
d_m = \frac{1}{\sqrt{2(2m + 3)^2(2m + 5), \ 0 \leq m \leq N - 4.}}
\]

Let \( l = (l_1, l_2) \) and

\[
\phi_l(x) = \chi_{l_1}(x_1) \chi_{l_2}(x_2).
\]

Then

\[
\Psi_{N}^{2,0} = \{ \phi_l(x) \mid 0 \leq l_1, l_2 \leq N - 4 \}.
\]

By the orthogonality of Legendre polynomials, both the matrix with the elements \( (\phi_l, \phi_{l'}) \) and the matrix with the elements \( (\nabla \phi_l, \nabla \phi_{l'}) \) are pentadiagonal. Hence this choice provides an efficient algorithm. In particular,

\[
\int_{-1}^{1} \frac{\partial^2 \chi_m(y) \partial y}{\partial y} \chi_{m'}(y) \, dy = \delta_{m, m'}
\]

which simplifies the calculation and improves the condition number of the related matrix essentially.

The numerical solution \( \psi_N(x, t) \) given by (5.25) is a discrete approximation to \( \Psi(x, t) \). Define two step functions \( \Psi_N^{m}(x, t) : [0, T] \to \Psi_{N}^{2,0}, m = 1, 2, \) such that for \( t \in \mathbb{R}_+(T), \)

\[
\Psi_N^{m}(x, t) = \psi_N(s + (2 - m)\tau), \quad t \in [s, s + \tau).
\]

Guo and He (1998) proved the following results.

**Theorem 5.6.** Let \( \Psi_0 \in H^0_0(\Omega), g \in C(0, T; L^2) \) and \( \psi_N_0 = \tilde{F}_N^{1,0} \Psi_0, \tau \leq \frac{C^*_N}{N^\tau}, C^*_N \) being a certain positive constant independent of \( N \) and \( \tau \). Then \( \Psi_N^{m}, m = 1, 2, \) converge to the weak solution \( \Psi \) in the following sense:

(i) \( \Psi_N^{m} \to \Psi \) in \( L^2(0, T; H^0_0(\Omega)) \) weakly, as \( N \to \infty, \tau \to 0; \)

(ii) \( \Psi_N^{m} \to \Psi \) in \( L^\infty(0, T; H^0_0(\Omega)) \) weak-star, as \( N \to \infty, \tau \to 0; \)

(iii) \( \Psi_N^{m} \to \Psi \) in \( L^2(0, T; H^0_0(\Omega)) \) strongly, as \( N \to \infty, \tau \to 0. \)
If in addition $\tau = o \left( N^{-4} \right)$, then

$$(iv) \ \Psi_N^{(m)} \to \Psi \text{ in } L^2(0, T; H_0^2(\Omega)) \text{ strongly, as } N \to \infty, \ \tau \to 0.$$
Then for all $t \in \tilde{R}_\tau(T)$,
\[
E(\Psi - \psi_N, t) \leq b_3 \left( \tau^2 + N^{2-2\tau_0} + N^{-2\tau_1} + \tau N^{2-2\tau_1} \right)
\]
where $b_3$ is a positive constant depending only on $\mu, \Omega$ and the norms of $\Psi$ and $g$ in the mentioned spaces. If in addition $\sigma > \frac{1}{2}$ and $\delta$ is suitably large, then the above result holds for any $\tau \to 0$.

Since we adopt partially implicit approximation for the nonlinear term in (5.25), the convergence rate in time $t$ is of first order. It limits the advantages of spectral approximations in the space. A modified version is the following
\[
\begin{align*}
(D_x \nabla \psi_N(t), \nabla \phi) + \frac{1}{4} J(\psi_N(t) + \psi_N(t + \tau), \psi_N(t) + \psi_N(t + \tau), \phi) + \frac{\mu}{2} (\Delta \psi_N(t) \\
+ \Delta \psi_N(t + \tau), \Delta \phi) + \frac{1}{2}(g(t) + g(t + \tau), \phi) = 0, \quad \forall \phi \in H^2_N, t \in \tilde{R}_\tau(T), \\
\psi_N(0) = P^2_0 \Psi_0, \quad x \in \tilde{\Omega}.
\end{align*}
\]
This scheme is of second order in time. But we have to solve a nonlinear equation at each time step. It is well known that if we use certain reasonable prediction-correction schemes for nonlinear problems, we could get the accuracy of second order in time and evaluate the values of unknown functions explicitly. He, Mao and Guo (1998) provided a prediction-correction scheme. Let $\psi_N^p(x, t)$ be the predicted value of $\psi_N(x, t)$. The prediction-correction scheme is as follows
\[
\begin{align*}
\frac{1}{\tau} \left( \nabla \left( \psi_N^{(p)}(t + \tau) - \psi_N(t) \right), \nabla \phi \right) + J(\psi_N(t), \psi_N(t), \phi) \\
+ \frac{\mu}{2} (\Delta \left( \psi_N^{(p)}(t + \tau) \right), \Delta \phi) + \frac{1}{2}(g(t), \phi) = 0, \quad \forall \phi \in H^2_N,
\end{align*}
\]
\[
\begin{align*}
\frac{1}{\tau} \left( \nabla (\psi_N(t + \tau) - \psi_N(t)), \nabla \phi \right) + \frac{1}{4} J(\psi_N(t), \psi_N(t), \phi) + \frac{1}{2} J \left( \psi_N^{(p)}(t + \tau), \psi_N^{(p)}(t + \tau), \phi \right) \\
+ \frac{\mu}{2} (\Delta (\psi_N(t + \tau) - \psi_N(t)), \Delta \phi) + \frac{1}{2}(g(t) + g(t + \tau), \phi) = 0, \quad \forall \phi \in H^2_N,
\end{align*}
\]
\[
\psi_N(0) = P^2_0 \Psi_0, \quad x \in \tilde{\Omega}.
\]
Let $\tau = O(N^{-4})$. It can be proved that if $\Psi \in L^2 \left( 0, T; H^{2\tau} \cap H^3_0 \right) \cap H^1 \left( 0, T; H^{2\tau} \cap \right) H^0 \left( 0, T; H^0 \right)$, $\Psi_0 \in H^2_0(\Omega)$ and $g \in H^2 \left( 0, T; L^2 \right)$, then there exists a positive constant $b_5$ depending only on $\mu, \Omega$ and the norms of $\Psi$ and $g$ in the mentioned spaces, such that for all $t \in \tilde{R}_\tau(T)$,
\[
E(\Psi - \psi_N, t) \leq b_5 \left( \tau^4 + N^{2-2\tau_0} + N^{-2\tau_1} + \tau N^{2-2\tau_1} \right).
\]

### 5.4. Spectral Domain Decomposition Methods

In this part, we consider spectral domain decomposition methods. The spatial computational domain is divided into several adjoining, nonintersecting subdomains, within
each of them we look for a polynomial solution. Some fulfillments are required on the common boundaries of adjoining subdomains. This approach stems from its capability of covering problems in complex geometry. Moreover, this approach also allows local refinement to resolve internal layers or discontinuities, maintaining however the spectral accuracy enjoyed by the standard spectral methods. In the previous parts, we already used the Fourier spectral and Fourier pseudospectral methods for an elliptic system, and the Legendre spectral method for a high order parabolic equation. In this part, we take a quasi-linear hyperbolic system as an example to describe the Chebyshev pseudospectral domain decomposition method.

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \). \( U \) and \( f \) are two vector functions with the components \( U^{(i)} \) and \( f^{(i)} \), \( 1 \leq j \leq p \). \( A_q(U) \) are \( p \times p \) matrices, \( 1 \leq q \leq n \). Consider the problem

\[
\begin{aligned}
\partial_t U + \sum_{q=1}^{n} A_q(U) \partial_q U &= f, & x \in \Omega, \ 0 < t \leq T,
U(x, 0) &= U_0(x), & x \in \bar{\Omega}
\end{aligned}
\]  
(5.28)

with suitable boundary conditions on \( \partial \Omega \times (0, T] \). For any \( \xi \in \mathbb{R}^n \) such that \( |\xi|_2 = 1 \), define the characteristic matrix for the direction \( \xi \) as

\[
A(\xi) = \sum_{q=1}^{n} A_q(U) \xi_q.
\]

(5.28) is assumed to be hyperbolic in time. It means that for any such \( \xi \), \( A(\xi) \) has \( p \) real eigenvalues and moreover it is diagonalizable. Let \( \lambda_k \) be the eigenvalues of \( A(\xi) \) and \( V_k \) be the corresponding left eigenvectors, \( 1 \leq k \leq p \). Suppose that \( \lambda_k > 0 \) for \( 1 \leq k \leq p_0 \) and \( \lambda_k < 0 \) for \( p_0 + 1 \leq k \leq p \). By taking the inner product of \( V_k \) with (5.28), we get that

\[
V_k \cdot \left( \partial_t U + \sum_{q=1}^{n} A_q(U) \partial_q U \right) = V_k \cdot f, \quad 1 \leq k \leq p.
\]

(5.29)

Denote by \( (\tau_1, \ldots, \tau_{n-1}) \) the system of Cartesian coordinates of the hyperplane orthogonal to the direction \( \xi \). Then for each \( h = 1, \ldots, n-1, \tau_h = (\tau_{h1}, \ldots, \tau_{hn}) \in \mathbb{R}^n, \tau_h \cdot \xi = 0 \) and \( |\tau_h| = 1 \). Owing to the identity

\[
\partial_q U = \xi_q \partial_q U + \sum_{h=1}^{n-1} \tau_{hq} \partial_{\tau_h} U,
\]

it follows from (5.29) that

\[
V_k \cdot \left( \partial_t U + \lambda_k \partial_q U \right) = V_k \cdot \left( f - \sum_{q=1}^{n} \left( A_q(U) \sum_{h=1}^{n-1} \tau_{hq} \partial_{\tau_h} U \right) \right), \quad 1 \leq k \leq p.
\]

(5.30)

We refer to (5.30) as the compatibility equations for (5.28).
Now, let $\Gamma$ be a $(n-1)$-dimensional manifold, and $\nu$ be the unit normal direction to $\Gamma$. Taking $\xi = \nu$, (5.30) restricted to $\Gamma$ becomes
\[
V_k \cdot (\partial_t U + \lambda_k \partial_n U) = V_k \cdot \left( f - \sum_{q=1}^{n} \left( A_q(U) \sum_{h=1}^{n-1} \tau_{h_q} \partial_{\tau_h} U \right) \right), \quad 1 \leq k \leq p. \quad (5.31)
\]

The right-hand side of (5.31) depends on the tangential derivatives of $U$ on $\Gamma$, while the left-hand side yields a combination of transport equations along a direction that is normal to $\Gamma$. If $\Gamma$ is the boundary $\partial \Omega$ and $\nu$ is oriented outward $\Omega$, then for $1 \leq k \leq p_0$, (5.31) yields $p_0$ transport equations according to which information are propagated from the inside to the outside of $\Omega$. These $p_0$ equations are called the compatibility equations for $\Omega$. On the other hand, if $\Gamma$ is the common boundary of the adjoining subdomains $\Omega_1$ and $\Omega_2$, taking as $\nu$ the normal direction to $\Gamma$, oriented from $\Omega_1$ to $\Omega_2$, then the first $p_0$ equations of (5.31) are the compatibility equations for $\Omega_1$. Obviously, for $p_0 + 1 \leq k \leq p$, (5.31) provides the compatibility equations for $\Omega_2$.

The use of compatibility equations for single domain pseudospectral methods was advocated in Gottlieb, Gunzburger and Turkel (1982). The issue of the compatibility equations at subdomain interfaces was addressed in Cambier, Ghazzi, Veuillot and Vivian (1982).

Now, let $\Omega$ be an $n$-dimensional hypercube, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma$, $\Omega_1 \cap \Omega_2 = \emptyset$ where $\Gamma$ is orthogonal to one Cartesian direction. We also assume for the moment that the solution $U$ is continuous across $\Gamma$. Let $N$ be any positive integer, and $\Sigma_N$ be the set of Chebyshev-Gauss-Lobatto interpolation points of the reference hypercube $[-1,1]^n$. Next, let $\Omega_{m,N}$ be the sets of the corresponding points in $\Omega_m$, and $\Omega_{m,N} = \bar{\Omega}_{m,N} \cap \Omega_m, m = 1,2$. Moreover, $\Gamma_N = \bar{\Omega}_{1,N} \cap \Gamma = \bar{\Omega}_{2,N} \cap \Gamma$, since we use the same number of interpolation points in $\bar{\Omega}_1$ and $\bar{\Omega}_2$.

Quarteroni (1990) served a Chebyshev pseudospectral domain decomposition scheme for (5.28). It is to look for $u_{1,N}(x,t) \in P_N$ for $x \in \Omega_1$ and $u_{2,N}(x,t) \in P_N$ for $x \in \Omega_2$, satisfying a set of equations. Firstly, they fulfill that
\[
\partial_t u_{1,N} + \sum_{q=1}^{n} A_q (u_{1,N}) \partial_q u_{1,N} = f, \quad x \in \Omega_{1,N}, \quad (5.32)
\]
\[
\partial_t u_{2,N} + \sum_{q=1}^{n} A_q (u_{2,N}) \partial_q u_{2,N} = f, \quad x \in \Omega_{2,N}, \quad (5.33)
\]
\[
u_{1,N} = u_{2,N}, \quad x \in \Gamma_N. \quad (5.34)
\]

In addition, we require that
\[
V_k \cdot (\partial_t u_{1,N} + \lambda_k \partial_n u_{1,N}) = V_k \cdot \left( f - \sum_{q=1}^{n} \left( A_q(u_{1,N}) \sum_{h=1}^{n-1} \tau_{h_q} \partial_{\tau_h} u_{1,N} \right) \right), \quad 1 \leq k \leq p_0, \quad (5.35)
\]
\[
V_k \cdot (\partial_t u_{2,N} + \lambda_k \partial_n u_{2,N}) = V_k \cdot \left( f - \sum_{q=1}^{n} \left( A_q(u_{2,N}) \sum_{h=1}^{n-1} \tau_{h_q} \partial_{\tau_h} u_{2,N} \right) \right), \quad p_0 + 1 \leq k \leq p. \quad (5.36)
\]
where \( \nu \) is the unit outward normal direction to \( \Omega_1 \) on \( \Gamma, \lambda^{(k)} \) and \( V^{(k)} \) are the eigenvalues and the left eigenvectors of the matrix \( A(\nu) \), and \( \tau \) is the tangential direction on \( \Gamma \). Indeed, (5.35) are the \( p_0 \) compatibility equations for \( \Omega_1 \) on \( \Gamma \), while (5.36) are the \( p - p_0 \) ones for \( \Omega_2 \) on \( \Gamma \). Furthermore at each point \( \bar{\Omega}_{1,N} \) belonging to a “face” \( \Psi \) of \( \partial \Omega_1 / \Gamma \), we set

\[
V_k \cdot (\partial_\nu u_{1,N} + \lambda_k \partial_\nu u_{1,N}) = V_k \cdot \left( f - \sum_{q=1}^{n} A_q (u_{1,N}) \sum_{h=1}^{n-1} \tau_h \partial_{\tau_h} u_{1,N} \right), \quad 1 \leq k \leq p_0,
\]

(5.37)

where \( \nu \) is now the outward normal direction to \( \Omega_1 \) on \( \Psi \), \( \tau \) is the tangential direction on \( \Psi \), while \( \lambda_k \) and \( V_k \) are the eigenvalues and the left eigenvectors of the matrix \( A(\nu) \). But here \( p_0 \) is not necessarily the same as in (5.35). The remaining \( p - p_0 \) equations that we need at each interpolation point of \( \Psi \) must be provided by the physical boundary conditions that supplement (5.28). Similarly, at each point \( \bar{\Omega}_{2,N} \) belonging to a “face” \( \Psi \) of \( \partial \Omega_2 / \Gamma \), we enforce that

\[
V_k \cdot (\partial_\nu u_{2,N} + \lambda_k \partial_\nu u_{2,N}) = V_k \cdot \left( f - \sum_{q=1}^{n} A_q (u_{2,N}) \sum_{h=1}^{n-1} \tau_h \partial_{\tau_h} u_{2,N} \right), \quad 1 \leq k \leq p_0,
\]

(5.38)

where \( \nu \) is the outward normal direction to \( \Omega_2 \) on \( \Psi \), \( \tau \) is the tangential direction on \( \Psi \), while \( \lambda_k \) and \( V_k \) are the eigenvalues and the left eigenvectors of the matrix \( A(\nu) \). But here \( p_0 \) is not necessarily the same as before. The remaining \( p - p_0 \) equations are prescribed as boundary conditions.

We now turn to the case in which \( \Gamma \) is no longer an arbitrary surface, rather the front of a shock propagating throughout \( \Omega \). Let \( \Omega_1 \) and \( \Omega_2 \) be the adjoining subdomains separated by \( \Gamma \), and \( w \) be the speed with which \( \Gamma \) propagates in the direction \( \nu, \nu \) being the outward normal vector to \( \Omega_1 \) on \( \Gamma \) as before. Assume that there exists a vector function \( F \) such that \( A_q (z) = \partial_z F_q (z) \). Then (5.28) can be written in conservation form as

\[
\partial_t U + \sum_{q=1}^{n} \partial_q F(U) = f, \quad x \in \Omega, \quad 0 < t \leq T.
\]

(5.39)

The weak solutions to (5.39) are allowed to be discontinuous across the interface \( \Gamma \), satisfying the Rankine-Hugoniot condition

\[
w[U] - \sum_{q=1}^{n} \nu_q \left[ F_q(U) \right] = 0
\]

where \( [z] \) denotes the jump of the values of \( z \) on the two sides of \( \Gamma \). In the context of the above Chebyshev pseudospectral approximation, the above equation reads as

\[
w (u_{2,N} - u_{1,N}) - \sum_{q=1}^{n} \nu_q (F_q (u_{2,N}) - F_q (u_{1,N})) = 0.
\]

(5.40)

At each interpolation point \( x_T \in \Gamma_{N1} \), (5.40) yields \( p \) equations for the \( 2p+1 \) unknowns which are \( u_{1,N}^{(j)}, u_{2,N}^{(j)}, 1 \leq j \leq p \) and \( w \). Thus we need \( p + 1 \) further independent
conditions provided by the compatibility, which, in the current situations, can be determined as follows. Let us define the characteristic matrix for $\Omega_1$ at the point on $\Gamma$, 

$$A\left(\nu^{(1)}\right) = \sum_{q=1}^{n} v_q A_q (u_{1,N})$$

and denote by $\lambda_k^{(1)}$ and $V_k^{(1)}$ the eigenvalues and the left eigenvectors respectively. Similarly denote by $\lambda_k^{(2)}$ and $V_k^{(2)}$ the eigenvalues and the left eigenvectors of the matrix 

$$A\left(\nu^{(2)}\right) = - \sum_{q=1}^{n} v_q A(u_{2,N}) .$$

In general, $u_{1,N} \neq u_{2,N}$ on $\Gamma$ and so $\lambda_k^{(1)} \neq -\lambda_k^{(2)} , V_k^{(1)} \neq V_k^{(2)}$. Assume that the eigenvalues are ordered as 

$$\lambda_k^{(1)} < \lambda_{k+1}^{(1)} , \quad \lambda_k^{(2)} < \lambda_{k+1}^{(2)} , \quad 1 \leq k \leq p-1$$

and that $\Gamma$ is the surface of propagation of a $k_1$-shock. So there exists an integer $1 \leq k_1 \leq p$ such that 

$$\lambda_{k_1-1}^{(1)} < w < \lambda_{k_1}^{(1)} , \quad \lambda_{k_1}^{(2)} < w < \lambda_{k_1+1}^{(2)} .$$

The compatibility equations for $\Omega_1$ are given by the following $p - k_1 + 1$ equations 

$$V_k^{(1)} \cdot \left( \partial_t u_{1,N} + \lambda_k^{(1)} \partial_{\nu^{(1)}} u_{1,N} \right) = V_k^{(1)} \cdot \left( f - \sum_{q=1}^{n} \left( A_q (u_{1,N}) \sum_{h=1}^{n-1} \tau_h \partial_{\tau_h} u_{1,N} \right) \right) , \quad k_1 \leq k \leq p ,$$

while those for $\Omega_2$ are given by the $k_1$ equations 

$$V_k^{(2)} \cdot \left( \partial_t u_{2,N} + \lambda_k^{(2)} \partial_{\nu^{(2)}} u_{2,N} \right) = V_k^{(2)} \cdot \left( f - \sum_{q=1}^{n} \left( A_q (u_{2,N}) \sum_{h=1}^{n-1} \tau_h \partial_{\tau_h} u_{2,N} \right) \right) , \quad p - k_1 + 1 \leq k \leq p .$$

As usual, $\{\tau_h\}$ is the set of vectors tangential to $\Gamma$ at the point $x_\Gamma$ under consideration.

We now consider the iteration by subdomain algorithm for the solution of the domain decomposition problem. For simplicity, assume that the solutions are continuous across the interface $\Gamma$. Let $D(\nu)$ and $E(\nu)$ denote the matrix of eigenvalues of $A(\nu)$ and the matrix of the corresponding left eigenvectors, so that 

$$E(\nu) A(\nu) = D(\nu) E(\nu) .$$

Let $u_{1,N,m}$ and $u_{2,N,m}$ be the $m$'th iterated values of $u_{1,N}$ and $u_{2,N}$ respectively. At each point $x \in \Gamma_N$, set 

$$\chi_{1,m} = u_{1,N,m} \quad \chi_{2,m} = u_{2,N,m} .$$
Moreover, we denote by \( E_{p_0} (\nu) \) the \( p_0 \times p \) matrix given by the first \( p_0 \) rows of \( E(\nu) \), and by \( E_{p-p_0} (\nu) \) the \( (p - p_0) \times p \) matrix given by the remaining \( p - p_0 \) rows of \( E(\nu) \). Then \( u_{1,N,m+1} \) satisfies the interior equations (5.32), the interface equation (5.35) together with
\[
E_{p-p_0} (\nu) u_{1,N,m+1} = E_{p-p_0} (\nu) \chi_{2,m}, \quad x \in \Gamma_N,
\]
and finally the boundary equations given by (5.37). Similarly, \( u_{2,N,m+1} \) satisfies the interior equations (5.33), the interface equation (5.36) together with
\[
E_{p_0} (\nu) u_{2,N,m+1} = E_{p_0} (\nu) \chi_{1,m}, \quad x \in \Gamma_N,
\]
and finally the boundary equations given by (5.38). It is noted that the limit solutions
\[
\lim_{m \to \infty} u_{1,N,m} \quad \text{and} \quad \lim_{m \to \infty} u_{2,N,m}
\]
fulfill the conditions
\[
E_{p_0} (\nu) u_{1,N} = E_{p_0} (\nu) u_{1,N}, \quad x \in \Gamma_N,
\]
\[
E_{p-p_0} (\nu) u_{1,N} = E_{p-p_0} (\nu) u_{2,N}, \quad x \in \Gamma_N,
\]
which in turn ensure the fulfillment of the continuity requirement (5.34).

We next consider a special case in which \( \Omega = (-1, 1), \Omega_1 = (-1, \alpha), \Omega_2 = (\alpha, 1), |\alpha| < 1 \). Clearly \( \Gamma = \{ \alpha \} \). The system (5.28) is reduced to
\[
\partial_t U + A(U) \partial_x U = f, \quad x \in \Omega, 0 < t \leq T,
\]
while the compatibility equations (5.30) become
\[
V_k \cdot (\partial_t U + \lambda_k \partial_x U) = V_k \cdot f, \quad 1 \leq k \leq p.
\]
Furthermore
\[
E \partial_t U + DE \partial_x U = Ef.
\]
Obviously, \( D \) and \( E \) depend on \( U \), if \( A \) does so.

Suppose that at the interface point \( x = \alpha \), the first \( p_0 \) eigenvalues of \( B \) are positive, and the others are negative. Let \( D_{p_0} \) and \( E_{p_0} \) denote the \( p_0 \times p \) matrices obtained suppressing the first \( p_0 \) rows of \( D \) and \( E \) respectively. Then the following \( p_0 \) equations
\[
E_{p_0} \partial_t U + D_{p_0} E \partial_x U = E_{p_0} f, \quad x = \alpha,
\]
provide the \( p_0 \) compatibility equations for \( \Omega_1 \). Similarly, denoting by \( D_{p-p_0} \) and \( E_{p-p_0} \) the lower part of the matrices \( D \) and \( E \), the equations
\[
E_{p-p_0} \partial_t U + D_{p-p_0} E \partial_x U = E_{p-p_0} f, \quad x = \alpha,
\]
provide the \( p - p_0 \) compatibility equations for \( \Omega_2 \). The compatibility equations for \( \Omega_2 \) at \( x = 1 \) are derived in a similar way, and take the form (5.43). At \( x = -1 \), the compatibility equations for \( \Omega_1 \) take the form (5.44).

The system (5.41) is completed by the initial condition
\[
U(x, 0) = U_0(x), \quad x \in \tilde{\Omega},
\]
and by a set of boundary conditions at \( x = \pm 1 \). For instance, \( B_1 \) is a \( p_0 \times p \) matrix and \( B_2 \) is a \( (p - p_0) \times p \) matrix, \( g_1 \) and \( g_2 \) are two given vector functions, and
\[
\begin{cases}
B_1 U = g_1, & \text{for } x = -1, \\
B_2 U = g_2, & \text{for } x = 1.
\end{cases}
\]
We now apply the Chebyshev pseudospectral domain decomposition method to (5.41). Let \( y^{(j)} = \cos \frac{\pi j}{N} \), the Chebyshev-Gauss-Lobatto interpolation points in the reference interval \([-1, 1]\), and
\[
 x_1^{(j)} = -1 + \frac{1 + \alpha}{2} \left( y^{(j)} + 1 \right), \quad x_2^{(j)} = \alpha + \frac{1 - \alpha}{2} \left( y^{(j)} + 1 \right), \quad 1 \leq j \leq N.
\]
At each time \( t \), we look for \( u_{1,N} \in \mathbb{P}_N \) for \( x \in \Omega_1 \) and \( u_{2,N} \in \mathbb{P}_N \) for \( x \in \Omega_2 \). They satisfy a set of equations as follows
\[
\begin{align*}
\partial_t u_{1,N} + A(u_{1,N}) \partial_x u_{1,N} &= f, \\
\partial_t u_{2,N} + A(u_{2,N}) \partial_x u_{2,N} &= f, \\
E_p \partial_t u_{1,N} + D_p E \partial_x u_{1,N} &= E_p f, \\
E_p \partial_t u_{2,N} + D_p E \partial_x u_{2,N} &= E_p f, \\
u_{1,N} &= u_{2,N}, \\
u_{1,N} &= u_{2,N}, \\
E_{p-p_0} \partial_t u_{1,N} + D_{p-p_0} E \partial_x u_{1,N} &= E_{p-p_0} f, \\
E_{p-p_0} \partial_t u_{2,N} + D_{p-p_0} E \partial_x u_{2,N} &= E_{p-p_0} f, \\
B_1 u_{1,N} &= g_1, \\
B_2 u_{2,N} &= g_2.
\end{align*}
\]
Obviously, the first two equations in the above set are the internal equations. The next three equations are the interface equations. The remaining ones are the boundary equations.

We now write the iteration. Let \( \chi_{1,m} = u_{1,N,m} \) and \( \chi_{2,m} = u_{2,N,m} \) at \( x = \alpha \).

Then we look for \( u_{1,N,m+1} \in \mathbb{P}_N \) in \( \Omega_1 \), such that
\[
\begin{align*}
\partial_t u_{1,N,m+1} + A \partial_x u_{1,N,m+1} &= f, \\
E_p \partial_t u_{1,N,m+1} + D_p E \partial_x u_{1,N,m+1} &= E_p f, \\
E_{p-p_0} u_{1,N,m+1} &= E_{p-p_0} \chi_{2,m}, \\
E_{p-p_0} \partial_t u_{1,N,m+1} + D_{p-p_0} E \partial_x u_{1,N,m+1} &= E_{p-p_0} f, \\
B_1 u_{1,N,m+1} &= g_1,
\end{align*}
\]
where \( A, D \) and \( E \) depend on \( u_{1,N,m+1} \). Similarly, we look for \( u_{2,N,m+1} \in \mathbb{P}_N \) in \( \Omega_2 \), such that
\[
\begin{align*}
\partial_t u_{2,N,m+1} + A \partial_x u_{2,N,m+1} &= f, \\
E_p \partial_t u_{2,N,m+1} + D_p E \partial_x u_{2,N,m+1} &= E_p f, \\
E_{p-p_0} u_{2,N,m+1} &= E_{p-p_0} \chi_{1,m}, \\
E_{p-p_0} \partial_t u_{2,N,m+1} + D_{p-p_0} E \partial_x u_{2,N,m+1} &= E_{p-p_0} f, \\
B_2 u_{2,N,m+1} &= g_2,
\end{align*}
\]
where \( A, D \) and \( E \) depend on \( u_{2,N,m+1} \).

Quarteroni (1990) proved the convergence of the above iteration in a special case.

Another early work of spectral domain decomposition methods for hyperbolic systems was due to Kopriva (1986). We also refer to Macaraeg, Streett and Hussaini (1989) for this technique and its applications. Recently Quarteroni, Pasquarelli and Valli (1991) developed heterogeneous domain decomposition method arising in the approximation of certain physical phenomena governed by different kinds of equations in several disjointed subregions. Various applications of domain decomposition methods can be found in Chan, Glowinski, Periaux and Widlund (1988), and Canuto, Hussaini, Quarteroni and Zang (1988).
5.5. Spectral Multigrid Methods

Multigrid methods began to be used for spectral approximations at the beginning of the last decade. Zang, Wong and Hussaini (1982, 1984) described this approach and its applications to periodic problems and non-periodic problems. It makes the spectral methods more efficient. In this part, we present some basic results in this topic.

We first let \( \Lambda = (0, 2\pi) \) and consider the model problem

\[
\begin{align*}
-\partial_x^2 U &= f, \\
U(x) &= U(x + 2\pi),
\end{align*}
\]

Let \( N \) be an even positive integer. Set \( x^{(j)} = \frac{2\pi j}{N} \) and \( f_j = f(x^{(j)}) \). Let \( u_j \) be the approximation to the value \( U(x^{(j)}), 0 \leq j \leq N - 1 \). Then

\[
u_j = \sum_{l=-\frac{N}{2}}^{\frac{N-1}{2}} \hat{u}_l \exp \left( \frac{2\pi ijl}{N} \right),
\]

\( \hat{u}_l \) being the discrete Fourier coefficients,

\[
\hat{u}_l = \frac{1}{N} \sum_{j=0}^{N-1} u_j \exp \left( -\frac{2\pi ijl}{N} \right), \quad l = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1.
\]

A sensible approximation to the left-hand side of (5.47) at \( x^{(j)} \) is

\[
\sum_{l=-\frac{N}{2}}^{\frac{N-1}{2}} l^2 \hat{u}_l \exp \left( \frac{2\pi ijl}{N} \right).
\]

Let \( V = (u_0, \ldots, u_{N-1}) \) and \( F = (f_0, \ldots, f_{N-1}) \). Moreover, we denote by \( D \) a diagonal matrix with the elements \( D_{jk} = j^2 \delta_{jk} \), and by \( C \) a matrix with the elements

\[
C_{jk} = \frac{1}{N} \exp \left( -\frac{2\pi ijk}{N} \right).
\]

Clearly

\[
(C^{-1})_{jk} = \exp \left( \frac{2\pi ijk}{N} \right).
\]

Set \( L = C^{-1}DC \). The pseudospectral approximation may be represented by

\[
LV = F.
\]

Let \( \omega \) be a relaxation parameter and \( V_m \) be the \( m \)th iterated value of \( V \). The Richardson iteration for (5.49) is

\[
V_{m+1} = V_m + \omega (F - LV_m).
\]
It can be checked that $L$ has the eigenvalues $\lambda_l = l^2$ and the corresponding eigenvectors $u_l$ with the components $u_l^{(j)} = \exp \left( \frac{2\pi i jl}{N} \right)$. Let $\tilde{V}_m = V_m - V$. It can be resolved into an expansion in the eigenvectors of $L$. Each iteration reduces the $l'$th error component by a factor $\nu(\lambda_l) = 1 - \omega \lambda_l$. Clearly the iteration is convergent, if for all $l$, $|1 - \omega \lambda_l| < 1$. In the current context, $\lambda_{\min} = 1$ and $\lambda_{\max} = \frac{1}{4} N^2$. So (5.50) is convergent for $0 < \omega < \frac{4}{N^2}$. The best choice of $\omega$ results from minimizing $|\nu(\lambda_l)|$ for $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. The optimal relaxation parameter for this single-grid procedure is

$$\omega_{SG} = \frac{2}{\lambda_{\max} + \lambda_{\min}}. \quad (5.51)$$

It produces the spectral radius

$$\rho_{SG} = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}.$$ 

Unfortunately, $\rho_{SG} \approx 1 - \frac{8}{N^4}$ which implies that $O(N^2)$ iterations are required to achieve the convergence. This slow convergence is the outcome of balancing the damping of the lowest frequency eigenfunctions with that of the highest frequency one in the minimax problem described in the above.

The multigrid approach takes advantage of the fact that the low-frequency modes ($|l| < \frac{N}{4}$) can be represented just as well on a coarser grid. So it settles for balancing the middle-frequency eigenfunction ($|l| = \frac{N}{4}$) with the highest frequency one ($|l| = \frac{N}{2}$). Namely, it damps effectively only those modes which can not be resolved on coarser grids. To do this, we replace $\lambda_{\min}$ in (5.51) by $\lambda_{\mid \mid}$, and obtain that

$$\omega_{MG} = \frac{2}{\lambda_{\max} + \lambda_{\mid \mid}}.$$

The multigrid smoothing factor

$$\mu_{MG} = \frac{\lambda_{\max} - \lambda_{\mid \mid}}{\lambda_{\max} + \lambda_{\mid \mid}},$$

measures the damping rate of the high frequency modes. In this example, $\lambda_{\mid \mid} = \frac{N^2}{16}$ and so $\mu_{MG} = 0.6$, independent of $N$.

We present the multigrid process by considering the interplay between two grids. The fine grid problem with $N_f$ interpolation points is written in the form

$$L_f V_f = F_f.$$ 

The decision to switch to the coarser grid is made after the fine grid approximation $W_f$ to $V_f$ has been sufficiently smoothed by the relaxation process, i.e., after the high frequency content of the error $\tilde{V}_f - W_f$ has been sufficiently reduced. Let $L_c$ be a coarse grid operator and $\tilde{V}_c$ be the correction of $W_f$ on the coarse grid with $N_c$ interpolation points. Furthermore denote by $R$ a restriction operator interpolating functions from the fine grid to the coarse grid, and

$$F_c = R (F_f - L_f \tilde{V}_f).$$
The auxiliary coarse grid problem is

\[ L_c V_c = F_c. \]

After an adequate approximation \( W_c \) to \( V_c \) has been obtained, the fine grid approximation is updated by

\[ W_f \leftarrow W_f + PW_c \]

where the prolongation operator \( P \) interpolates functions from the coarse grid to the fine grid. We can repeat the above procedure with a nest of grids and a control structure to reach a desired result.

We now construct the restriction operator \( R \) and the prolongation operator \( P \). Consider the problem

\[ \partial_x (a(x) \partial_x U(x)) = f(x), \quad x \in \Lambda \quad (5.52) \]

where \( \Lambda = (0, 2\pi) \) in the periodic case or \( \Lambda = (-1, 1) \) in the Dirichlet case.

We first consider the periodic case. On the coarse grid, the discrete Fourier coefficients of the corrections \( u_j \), at the coarse grid interpolation points \( \bar{x}^{(j)} \) are computed using (5.48) with \( N = N_c \). The fine grid approximation is then updated by

\[ u_j \leftarrow u_j + \sum_{l=0}^{N_c-1} \tilde{u}_l \exp \left( il \bar{x}^{(j)} \right), \quad 0 \leq j \leq N_f - 1, \]

where \( \bar{x}^{(j)} \) are the fine grid interpolation points. The restriction operator is constructed in a similar fashion. It turns out that except for a factor of \( \frac{N_f}{N_c} \), \( P \) and \( R \) are adjoint. An explicit representation of the prolongation operator is

\[ P_{jk} = \frac{1}{N_c} \sum_{l=1}^{N_c-1} \exp \left( 2\pi il \left( \frac{j}{N_f} - \frac{k}{N_c} \right) \right). \]

It sums to yield

\[ P_{jk} = \frac{1}{N_c} \delta\left( \frac{j}{N_f} - \frac{k}{N_c} \right) \]

where

\[ S(r) = \begin{cases} N_c - 1, & r \text{ is an integer}, \\ \sin(\pi r N_c) \csc(\pi r) - \cos(\pi r N_c), & \text{otherwise}. \end{cases} \]

The corresponding restriction operator is

\[ R_{jk} = \frac{1}{N_f} S\left( \frac{j}{N_c} - \frac{k}{N_f} \right). \]

We next consider the Dirichlet case, and use Chebyshev approximation. The prolongation operator is

\[ P_{jk} = \frac{2}{\xi_k N_c} \sum_{l=0}^{N_c} \xi^{-1}_l \cos \frac{l_j}{N_f} \cos \frac{l_k}{N_c} \]
where $\tilde{c}_k = 2$ for $k = 0, N_c$ and $\tilde{c}_k = 1$ for $1 \leq k \leq N_c - 1$. This sums to

$$P_{jk} = \frac{2}{\tilde{c}_k N_c} \left( Q \left( \frac{j}{N_f} - \frac{k}{N_f} \right) + Q \left( \frac{j}{N_c} + \frac{k}{N_c} \right) \right)$$

where

$$Q(r) = \begin{cases} \frac{N_c}{2}, & r \text{ is an integer}, \\ \frac{1}{4} - \frac{1}{4} \cos \left( \frac{\pi r N_c}{2} \right), & \text{otherwise}. \end{cases}$$

One kind of restriction operator is defined as

$$R_{jk} = \frac{2}{\tilde{c}_k N_f} \left( Q \left( \frac{j}{N_c} - \frac{k}{N_f} \right) + Q \left( \frac{j}{N_c} + \frac{k}{N_f} \right) \right)$$

where $\tilde{c}_k = 2$ for $k = 0, N_f$ and $\tilde{c}_k = 1$ otherwise, and

$$Q(r) = \begin{cases} \frac{1}{4} + \frac{N_c}{2}, & r \text{ is an integer}, \\ \frac{1}{4} + \frac{1}{2} \cos \left( \frac{\pi r (N_c + 1)}{2} \right) \sin \left( \frac{\pi r N_c}{2} \right) \csc \left( \frac{\pi r}{2} \right), & \text{otherwise}. \end{cases}$$

The other is defined by the adjoint requirement, i.e.,

$$R_{jk} = \frac{1}{\tilde{c}_k N_c} \left( Q \left( \frac{j}{N_c} - \frac{k}{N_f} \right) + Q \left( \frac{j}{N_c} + \frac{k}{N_f} \right) \right).$$

For the problem (5.52), the discrete operator on the fine grid is

$$L_f = D_{N_f} A D_{N_f}$$

where $A$ is the diagonal matrix with the elements $A_{jk} = a(x^{(j)}) \delta_{j,k}$, $D_{N_f}$ is the matrix related to the differentiation. In the periodic case,

$$(D_{N_f})_{jk} = \begin{cases} \frac{1}{2} (-1)^{j+k} \cot \left( \frac{\pi}{N_f} (j - k) \right), & j \neq k, \\ 0, & j = k. \end{cases}$$

In the Dirichlet case,

$$(D_{N_f})_{jk} = \begin{cases} \frac{\tilde{c}_j (-1)^{j+k}}{\tilde{c}_k (x^{(j)} - x^{(k)})}, & j \neq k, \\ \frac{2}{6}, & 1 \leq j = k \leq N_f - 1, \\ \frac{2 N_f^2 + 1}{6}, & j = k = 1, \\ \frac{2 N_f^2 + 1}{6}, & j = k = N_f. \end{cases}$$
We now turn to two-dimensional problems. Let $\Omega = (-1, 1)^2$ and consider the self-adjoint elliptic equation

$$
\partial_1 (a_1(x) \partial_1 U(x)) + \partial_2 (a_2(x) \partial_2 U(x)) = f(x), \quad x \in (-1, 1)^2
$$

(5.53)

with the Dirichlet boundary condition. We use pseudospectral method with $(N+1)^2$ Chebyshev-Gauss-Lobatto interpolation points. It leads to a discrete set of equations like (5.49). The convergence rate of Richardson iteration on a single grid is governed by the ratio of the largest eigenvalue to the smallest eigenvalue of $L$, referred to as the condition number. The multigrid condition number is the ratio of the largest eigenvalue to the smallest high frequency eigenvalue. In our current context, $\lambda_{\text{max}} = O(N^4)$, $\lambda_{\text{mid}} = O(N^2)$ and $\lambda_{\text{min}} \approx \frac{2}{N}$. An effective preconditioning is essential for multigrid as well as for single-grid iterative schemes.

The preconditioned Richardson iteration can be expressed as

$$
V \leftarrow V + \omega H^{-1}(F - LV)
$$

where $H$ is a preconditioning matrix. An obvious choice of $H$ is a finite difference approximation $H_{FD}$ to the differential operator in (5.53). But for multi-dimensional problems, these finite difference approximations are costly to invert. An attractive alternative is to use an approximate $LU$-decomposition. In this case, $H$ is taken as the product of a lower triangular matrix $L$ and an upper triangular matrix $U$. Buleev (1960) and Oliphant (1961) proposed one such type of preconditioning by taking $L$ as the lower triangular portion of $H_{FD}$ and choosing $U$ such that the two super diagonals of $UL$ agree with those of $H_{FD}$, see Zang, Wong and Hussaini (1984). Zang, Wong and Hussaini (1982) also gave a similar decomposition denoted by $H_{RS}$, in which the diagonal elements of $L$ are altered from those of $H_{FD}$ to ensure that the row sums of $H_{RS}$ and $H_{FD}$ are identical. Both types of preconditioning can be computed by a simple recursion. Let $u_{j,k}$ be the values of $u$ at the interior grid point $(x_j^{(1)}, x_k^{(1)})$. Suppose that a five point finite difference approximation to (5.53) at this point is given by

$$
e_{j,k} u_{j,k} + d_{j,k} u_{j-1,k} + b_{j,k} u_{j,k-1} + g_{j,k} u_{j+1,k} + h_{j,k} u_{j,k+1} = f_{j,k},
$$

and that a five-diagonal incomplete $LU$ factorization is given by

$$(LU)_{j,k} = u_{j,k} + \beta_{j,k} u_{j-1,k} + \gamma_{j,k} u_{j,k-1},$$

$$(UL)_{j,k} = u_{j,k} + \lambda_{j,k} u_{j+1,k} + \sigma_{j,k} u_{j,k+1}.$$ 

Set

$$
\beta_{j,k} = d_{j,k},
\gamma_{j,k} = b_{j,k},
\nu_{j,k} = e_{j,k} - \beta_{j,k} \sigma_{j,k-1} - \gamma_{j,k} \lambda_{j-1,k} - \alpha (\beta_{j,k} \lambda_{j,k-1} + \gamma_{j,k} \sigma_{j-1,k}),
\lambda_{j,k} = \frac{g_{j,k}}{\nu_{j,k}},
\sigma_{j,k} = \frac{h_{j,k}}{\nu_{j,k}}.
$$
Then $\alpha = 0$ gives $H_{LU}$ and $H_{RS}$ uses $\alpha = 1$. Straightforward modifications are made near the boundaries. A more accurate factorization can be achieved by including one extra nonzero diagonal in $L$ and $U$. Suppose that
\[
(LV)_{j,k} = \nu_{j,k} u_{j,k} + \beta_{j,k} u_{j-1,k} + \gamma_{j,k} u_{j,k-1} + \theta_{j,k} u_{i+1,j-1},
\]
\[
(UV)_{j,k} = u_{j,k} + \lambda_{j,k} u_{j+1,k} + \sigma_{j,k} u_{j,k+1} + \chi_{j,k} u_{i-1,j+1} .
\]

Put
\[
\gamma_{j,k} = b_{j,k},
\]
\[
\theta_{j,k} = -\gamma_{j,k} \lambda_{j,k-1},
\]
\[
\beta_{j,k} = d_{j,k} - \gamma_{j,k} \chi_{j,k-1},
\]
\[
\nu_{j,k} = e_{j,k} - \gamma_{j,k} \sigma_{j,k-1} - \theta_{j,k} \chi_{j+1,k-1} - \beta_{j,k} \lambda_{j-1,k} - \alpha (\lambda_{j+1,k-1} \theta_{j,k} - \chi_{j-1,k} \beta_{j,k}),
\]
\[
\lambda_{j,k} = \frac{1}{\nu_{j,k}} (g_{j,k} - \sigma_{j+1,k-1} \theta_{j,k}),
\]
\[
\chi_{j,k} = \frac{-\beta_{j,k} \sigma_{j-1,k}}{\nu_{j,k}},
\]
\[
\sigma_{j,k} = \frac{h_{j,k}}{\nu_{j,k}} .
\]

Once again, $\alpha = 0$ gives the $H_{LU}$ version and $\alpha = 1$ gives the $H_{RS}$ version.

Spectral multigrid methods have been used for nonlinear problems, such as in Zang and Hussaini (1986).
Lecture 6

Mixed Spectral Methods

In studying boundary layer, channel flow, flow past a suddenly heated vertical plate and some other problems, we meet semi-periodic problems. It is reasonable to deal with them using Fourier approximations in the periodic directions. We can use finite difference approximations or finite element approximations in the non-periodic directions. But in these cases, the accuracy in space is still limited due to these approximations, even if the genuine solution of an original problem is infinitely smooth. This deficiency can be removed by mixed Fourier-Legendre approximations or mixed Fourier-Chebyshev approximations.

6.1. Mixed Fourier-Legendre Approximations

In this part, we introduce the mixed Fourier-Legendre approximations. Let \( x = (x_1, x_2), \Lambda_1 = \{x_1 \mid -1 < x_1 < 1\}, \Lambda_2 = \{x_2 \mid 0 < x_2 < 2\pi\} \) and \( \Omega = \Lambda_1 \times \Lambda_2 \). For describing the approximation results by using different orthogonal systems in different directions, we define some non-isotropic spaces as follows,

\[
H^{r,s}(\Omega) = L^2(\Lambda_2, H^r(\Lambda_1)) \cap H^s(\Lambda_2, L^2(\Lambda_1)), \quad r, s \geq 0,
\]

\[
M^{r,s}(\Omega) = H^{r,s}(\Omega) \cap H^1(\Lambda_2, H^{r-1}(\Lambda_1)) \cap H^{s-1}(\Lambda_2, H^1(\Lambda_1)), \quad r, s \geq 1,
\]

\[
X^{r,s}(\Omega) = H^s(\Lambda_2, H^{r+1}(\Lambda_1)) \cap H^{s+1}(\Lambda_2, H^r(\Lambda_1)), \quad r, s \geq 0
\]

equipped with the norms

\[
\|v\|_{H^{r,s}(\Omega)} = \left( \|v\|_{L^2(\Lambda_2, H^r(\Lambda_1))}^2 + \|v\|_{H^s(\Lambda_2, L^2(\Lambda_1))}^2 \right)^{\frac{1}{2}},
\]

\[
\|v\|_{M^{r,s}(\Omega)} = \left( \|v\|_{H^{r,s}(\Omega)}^2 + \|v\|_{H^1(\Lambda_2, H^{r-1}(\Lambda_1))}^2 + \|v\|_{H^{s-1}(\Lambda_2, H^1(\Lambda_1))}^2 \right)^{\frac{1}{2}},
\]

\[
\|v\|_{X^{r,s}(\Omega)} = \left( \|v\|_{H^s(\Lambda_2, H^{r+1}(\Lambda_1))}^2 + \|v\|_{H^{s+1}(\Lambda_2, H^r(\Lambda_1))}^2 \right)^{\frac{1}{2}}.
\]

Denote by \( \bar{D}_p(\Omega) \) the set of all infinitely differentiable functions with compact supports in \( \Lambda_1 \) and the period \( 2\pi \) for the variable \( x_2 \). The closures of \( \bar{D}_p(\Omega) \) in \( H^{r,s}(\Omega), M^{r,s}(\Omega) \) and \( X^{r,s}(\Omega) \) are denoted by \( H^{r,s}_0(\Omega), M^{r,s}_0(\Omega) \) and \( X^{r,s}_0(\Omega) \), respectively. For the sake of simplicity, we denote the norms \( \|\cdot\|_{H^{r,s}(\Omega)}, \|\cdot\|_{M^{r,s}(\Omega)} \) and \( \|\cdot\|_{X^{r,s}(\Omega)} \) also by \( \|\cdot\|_{H^{r,s}}, \|\cdot\|_{M^{r,s}} \) and \( \|\cdot\|_{X^{r,s}} \), respectively. Furthermore, \( \|\cdot\|_{H^{r,s}} \) and \( \|\cdot\|_{M^{r,s}} \) stand for the semi-norms, associated with the norm \( \|\cdot\|_{H^{r,s}} \) and \( \|\cdot\|_{M^{r,s}} \), respectively. Besides, \( H^{r,s}_0(\Omega) \) denotes the closure in \( H^{r,s}(\Omega) \) of the set involving all infinitely differentiable functions with the period \( 2\pi \) for the variable \( x_2 \), etc..
Let $M$ and $N$ be any positive integers. Denote by $\mathbb{P}_{M,1}$ the set of all polynomials of degree at most $M$ on the interval $\Lambda_1$ and $\mathbb{P}_{M,1}^0 = \mathbb{P}_{M,1} \cap H_0^1 ( \Lambda_1 )$. Denote by $\tilde{V}_{N,2}$ the set of all trigonometric polynomials of degree at most $N$ on the interval $\Lambda_2$, and by $V_{N,2}$ the subset of $\tilde{V}_{N,2}$ containing all real-valued functions. Set $V_{M,N} = \mathbb{P}_{M,1} \otimes V_{N,2}$ and $V_{M,N}^0 = \mathbb{P}_{M,1}^0 \otimes V_{N,2}$. In the space $V_{M,N}$, several inverse inequalities are valid.

**Theorem 6.1.** For any $\phi \in V_{M,N}$ and $2 \leq p \leq \infty$,

$$\| \phi \|_{L^p} \leq c \left( M^2 N \right)^{\frac{1}{p} - \frac{1}{2}} \| \phi \|.$$  

**Theorem 6.2.** For any $\phi \in V_{M,N}$,

$$| \phi |^2 \leq \left( \frac{9}{4} M^4 + N^2 \right) \| \phi \|^2.$$  

The mixed Fourier-Legendre expansion of a function $v \in L^2(\Omega)$ is

$$v(x) = \sum_{m=0}^{\infty} \sum_{|l|=0}^{\infty} \hat{v}_{m,l} L_m (x_1) e^{ikx}$$

where $\hat{v}_{m,l}$ is the Fourier-Legendre coefficient,

$$\hat{v}_{m,l} = \frac{1}{2\pi} \left( m + \frac{1}{2} \right) \int_{\Omega} v(x) L_m (x_1) e^{-ikx} dx.$$  

The $L^2(\Omega)$-orthogonal projection $P_{M,N} : L_p^2(\Omega) \to V_{M,N}$ is such a mapping that for any $v \in L_p^2(\Omega)$,

$$(v - P_{M,N} v, \phi) = 0, \quad \forall \phi \in V_{M,N}.$$  

**Theorem 6.3.** For any $v \in H_p^{r,s}(\Omega)$ and $r, s \geq 0$,

$$\| v - P_{M,N} v \| \leq c \left( M^{-r} + N^{-s} \right) \| v \|_{H^r \cap \cap H^s}.$$  

It is noted that for any $v \in H_0^{r,s}(\Omega)$ and $r, s \geq 0$,

$$\| v - P_{M,N} v \| \leq c \left( M^{-r} + N^{-s} \right) \| v \|_{H^r \cap \cap H^s}.$$  

When we use the mixed Fourier-Legendre approximations for semi-periodic problems of partial differential equations, we need different kinds of orthogonal projections to obtain the optimal error estimates. For instance, the $H^1(\Omega)$-orthogonal projection $P_{M,N} : H^1_0(\Omega) \to V_{M,N}$ is such a mapping that for any $v \in H^1_0(\Omega)$,

$$\left( \nabla (v - P_{M,N} v), \nabla \phi \right) + (v, \phi) = 0, \quad \forall \phi \in V_{M,N}.$$  

While the $H^1_0(\Omega)$-orthogonal projection $P_{M,N}^0 : H^1_0(\Omega) \to V_{M,N}^0$ is such a mapping that for any $v \in H^1_0(\Omega)$,

$$\left( \nabla (v - P_{M,N}^0 v), \nabla \phi \right) = 0, \quad \forall \phi \in V_{M,N}^0.$$
Theorem 6.4. If \( v \in M_\mu^r s(\Omega) \) with \( r, s \geq 1 \), then for \( \mu = 0, 1 \),
\[
\| v - P_{M,N}^1 v \|_\mu \leq c \left( M^{1-r} + N^{1-s} \right) \left( M^{\mu-1} + N^{\mu-1} \right) \| v \|_{M^s \mu}.
\]
If, in addition, for certain positive constants \( c_1 \) and \( c_2 \),
\[
c_1 N \leq M \leq c_2 N,
\]
then
\[
\| v - P_{M,N}^1 v \| \leq c \left( M^{1-r} + N^{-s} \right) \| v \|_{M^s \mu}.
\]

Theorem 6.5. If \( v \in H_{0,p}^1(\Omega) \cap M_\mu^r s(\Omega) \) with \( r, s \geq 1 \), then for \( \mu = 0, 1 \),
\[
\| v - P_{M,N}^{1,0} v \|_\mu \leq c \left( M^{1-r} + N^{1-s} \right) \left( M^{\mu-1} + N^{\mu-1} \right) \| v \|_{M^s \mu}.
\]
If in addition (6.1) holds, then
\[
\| v - P_{M,N}^{1,0} v \| \leq c \left( M^{1-r} + N^{-s} \right) \| v \|_{M^s \mu}.
\]

In numerical analysis of mixed Fourier-Legendre spectral schemes for nonlinear problems, we also need to bound the norm \( \| P_{M,N}^{1,0} v \|_{W^r \mu} \). One of them is stated in the following theorem.

Theorem 6.6. Let (6.1) hold and \( s > \frac{1}{2} \). If \( v \in H_{0,p}^1(\Omega) \cap H_p^s(\Lambda_2, H^1(\Lambda_1)) \), then
\[
\| P_{M,N}^{1,0} v \|_{L^\infty} \leq c \| v \|_{H^s(\Lambda_2, H^1(\Lambda_1))}.
\]
If, in addition, \( v \in X_{p,p}^r(\Omega) \) with \( r \geq 1 \), then
\[
\| P_{M,N}^{1,0} v \|_{W^{1,r}} \leq c \| v \|_{X_{p,p}^r}.
\]

Finally we consider the mixed Fourier-Legendre interpolation. Let \( I_M \) be the same as in the previous paragraphs and \( I_N \) be the Fourier interpolation on the interval \( \Lambda_2 \) respectively. Set \( I_{M,N} = I_M I_N = I_N I_M \).

Theorem 6.7. If \( v \in H^\beta_p(\Lambda_2, H^r(\Lambda_1)) \cap H_p^s(\Lambda_2, H^\alpha(\Lambda_1)) \cap H^\beta_p(\Lambda_2, H^\gamma(\Lambda_1)) \), \( 0 \leq \alpha \leq \min(r, \lambda), 0 \leq \beta \leq \min(s, \gamma) \) and \( r, s, \lambda, \gamma > \frac{1}{2} \), then
\[
\| v - I_{M,N} v \|_{H^s(\Lambda_2, H^{\alpha}(\Lambda_1))} \leq c M^{2\alpha-r+\frac{1}{2}} \| v \|_{H^s(\Lambda_2, H^r(\Lambda_1))} + c N^{\gamma-\frac{1}{2}} \| v \|_{H^s(\Lambda_2, H^{\alpha}(\Lambda_1))} + c q(\beta) M^{2\alpha-\lambda+\frac{1}{2}} N^{\beta-\gamma} \| v \|_{H^r(\Lambda_2, H^{\lambda}(\Lambda_1))}
\]
where \( q(\beta) = 0 \) for \( \beta > \frac{1}{2} \) and \( q(\beta) = 1 \) for \( \beta \leq \frac{1}{2} \).

It is noted that if \( 0 \leq \alpha \leq 1 \) and \( \alpha < \min(2r-1, 2\lambda - 1) \), then
\[
\| v - I_{M,N} v \|_{H^s(\Lambda_2, H^{\alpha}(\Lambda_1))} \leq c M^{2\alpha-r} \| v \|_{H^s(\Lambda_2, H^{\alpha}(\Lambda_1))} + c N^{\gamma-\frac{1}{2}} \| v \|_{H^s(\Lambda_2, H^{\alpha}(\Lambda_1))} + c q(\beta) M^{\alpha-\lambda} N^{\beta-\gamma} \| v \|_{H^{r}(\Lambda_2, H^{\lambda}(\Lambda_1))},
\]
where \( q(\beta) \) being the same as in Theorem 6.7.

Theorem 6.3 and Theorem 6.5 can be found in Bernardi, Maday and Metivet (1987). The others can be found in Guo (1998).
6.2. Mixed Fourier-Chebyshev Approximations

This part is devoted to the mixed Fourier-Chebyshev approximations. Let $\Lambda_1, \Lambda_2$ and $\Omega$ be the same as in the previous part. Set $\omega(x) = \omega(x_1) = (1 - x_1^2)^{-\frac{1}{2}}$, and

$$(v, w)_\omega = \int_\Omega v(x)w(x)\omega(x)\, dx, \quad ||v||_\omega = (v, v)_\omega^{\frac{1}{2}}.$$

Define

$$L^2_\omega(\Omega) = \{v \mid v \text{ is measurable, } ||v||_\omega < \infty\}.$$

We define the spaces $L^r_\omega(\Omega), H^s_\omega(\Omega), W^{r,s}_\omega(\Omega)$ and their norms in a similar way. We also introduce some non-isotropic spaces as follows

$$H^{r,s}_\omega(\Omega) = L^2(\Lambda_2, H^{r,s}_\omega(\Lambda_1)) \cap H^s(\Lambda_2, L^2_\omega(\Lambda_1), \quad r, s \geq 0,$$

$$M^{r,s}_\omega(\Omega) = H^{r,s}_\omega(\Omega) \cap H^1(\Lambda_2, H^{r,s-1}_\omega(\Lambda_1)), \quad r, s \geq 1,$$

$$X^{r,s}_\omega(\Omega) = H^s(\Lambda_2, H^{r,s+1}_\omega(\Lambda_1)) \cap H^{r+1}(\Lambda_2, H^1_\omega(\Lambda_1)), \quad r, s \geq 0$$

equipped with the norms

$$||v||_{H^{r,s}_\omega(\Omega)} = \left(||v||^2_{L^2(\Lambda_2, H^{r,s}_\omega(\Lambda_1))} + ||v||^2_{H^s(\Lambda_2, L^2_\omega(\Lambda_1))}\right)^{\frac{1}{2}},$$

$$||v||_{M^{r,s}_\omega(\Omega)} = \left(||v||^2_{H^{r,s}_\omega(\Omega)} + ||v||^2_{H^1(\Lambda_2, H^{r,s-1}_\omega(\Lambda_1))} + ||v||^2_{H^{r,s-1}(\Lambda_2, H^1_\omega(\Lambda_1))}\right)^{\frac{1}{2}},$$

$$||v||_{X^{r,s}_\omega(\Omega)} = \left(||v||^2_{H^s(\Lambda_2, H^{r,s+1}_\omega(\Lambda_1))} + ||v||^2_{H^{r+1}(\Lambda_2, H^1_\omega(\Lambda_1))}\right)^{\frac{1}{2}}.$$

The meaning of $\overline{D}_p(\Omega)$ is the same as in the previous section. The closures of $\overline{D}_p(\Omega)$ in $H^{r,s}_\omega(\Omega), M^{r,s}_\omega(\Omega), X^{r,s}_\omega(\Omega)$ are denoted by $H^{r,s}_0(\Omega), M^{r,s}_0(\Omega), \overline{X}^{r,s}_\omega(\Omega)$ respectively. Their norms are denoted also by $||\cdot||_{H^{r,s}_0}, ||\cdot||_{M^{r,s}_0}, ||\cdot||_{\overline{X}^{r,s}_\omega}$ for simplicity. The corresponding semi-norms are denoted by $|\cdot|_{H^{r,s}_0}, |\cdot|_{M^{r,s}_0}, |\cdot|_{\overline{X}^{r,s}_\omega}$. The space $H^{r,s}_p(\Omega)$ is the closure in $H^{r,s}_\omega(\Omega)$ of the set containing all infinitely differentiable functions with period $2\pi$ in the $x_2$-direction, etc.

Let $M$ and $N$ be any positive integers. The subspaces $P_{M,1}, P_{M,1}^0, V_{N,2}, V_{N,2}, V_{M,N}$ and $V_{M,N}$ have the same meanings as in the previous part. We first give some inverse inequalities.

**Theorem 6.8.** For any $\phi \in V_{M,N}$ and $2 \leq p \leq \infty$,

$$||\phi||_{L^p_\omega} \leq c(MN)^{\frac{1}{2} - \frac{1}{p}} ||\phi||_\omega.$$

As a consequence of Theorem 6.8, we have that for any $\phi, \psi \in V_{M,N}$,

$$||\phi\psi||^2_\omega \leq cMN||\phi||^2_\omega||\psi||^2_\omega.$$

**Theorem 6.9.** For any $\phi \in V_{M,N}$,

$$||\phi||^{2}_{L^\infty_{\omega}} \leq (4M^4 + N^2) ||\phi||^{2}_{L^\infty_\omega}. $$
The mixed Fourier-Chebyshev expansion of a function \( v \in L^2_\omega(\Omega) \) is
\[
v(x) = \sum_{m=0}^{\infty} \sum_{|l|=0}^{\infty} \hat{v}_{m,l} T_m(x_1) e^{ilx_2}
\]
with the Fourier-Chebyshev coefficients
\[
\hat{v}_{m,l} = \frac{1}{\pi^2 c_l} \int_\Omega v(x) T_m(x_1) e^{ilx_2} \, dx.
\]
The \( L^2(\Omega) \)-orthogonal projection \( P_{M,N} : L^2_\omega(\Omega) \to V_{M,N} \) is such a mapping that for any \( v \in L^2_\omega(\Omega) \),
\[
(v - P_{M,N}v, \phi)_\omega = 0, \quad \forall \, \phi \in V_{M,N}.
\]

**Theorem 6.10.** For any \( v \in H^{r,s}_{p,\omega}(\Omega) \) and \( r, s \geq 0 \),
\[
||v - P_{M,N}v||_{H^{r,s}} \leq c \left( M^{-r} + N^{-s} \right) ||v||_{H^{r,s}}.
\]
It is noted that for any \( v \in H^{r,s}_{0,\omega}(\Omega) \) and \( r, s \geq 0 \),
\[
||v - P_{M,N}v||_{\omega} \leq c \left( M^{-r} + N^{-s} \right) ||v||_{H^{r,s}}.
\]
When we apply the mixed Fourier-Chebyshev approximations to different semi-periodic problems of partial differential equations, we adopt different orthogonal projections to obtain the optimal error estimations. For instance, the \( H^{1}_\omega(\Omega) \)-orthogonal projection \( P_{M,N}^{1} : H^{1}_\omega(\Omega) \to V_{M,N}^{1} \) is such a mapping that for any \( v \in H^{1}_\omega(\Omega) \),
\[
\left( \nabla(v - P_{M,N}^{1}v), \nabla \phi \right)_\omega + (v, \phi)_\omega = 0, \quad \forall \, \phi \in V_{M,N}^{1}.
\]
While the \( H^{1,0}_\omega(\Omega) \)-orthogonal projection \( P_{M,N}^{1,0} : H^{1,0}_\omega(\Omega) \to V_{M,N}^{0} \) is such a mapping that for any \( v \in H^{1,0}_\omega(\Omega) \),
\[
\left( \nabla(v - P_{M,N}^{1,0}v), \nabla \phi \right) = 0, \quad \forall \, \phi \in V_{M,N}^{0}.
\]

**Theorem 6.11.** If \( v \in M^{r,s}_{p,\omega}(\Omega) \) with \( r, s \geq 1 \), then for \( \mu = 0,1 \),
\[
||v - P_{M,N}^{1,0}v||_{\mu, \omega} \leq c \left( M^{1-r} + N^{1-s} \right) \left( M^{\mu-1} + N^{\mu-1} \right) ||v||_{M^{\mu,s}}.
\]
If in addition (6.1) holds, then
\[
||v - P_{M,N}^{1,0}v||_{\omega} \leq c \left( M^{-r} + N^{-s} \right) ||v||_{M^{\mu,s}}.
\]

**Theorem 6.12.** If \( v \in H^{1}_{0,\omega}(\Omega) \cap M^{r,s}_{p,\omega}(\Omega) \) with \( r, s \geq 1 \), then
\[
||v - P_{M,N}^{1,0}v||_{\mu, \omega} \leq c \left( M^{1-r} + N^{1-s} \right) \left( M^{\mu-1} + N^{\mu-1} \right) ||v||_{M^{\mu,s}}, \quad \mu = 0,1.
\]
If in addition (6.1) holds, then
\[
||v - P_{M,N}^{1,0}v||_{\omega} \leq c \left( M^{-r} + N^{-s} \right) ||v||_{M^{\mu,s}}.
\]
In numerical analysis of mixed Fourier-Chebyshev spectral schemes for nonlinear problems, we have to estimate the norm \( \| P_{M,N}^1 v \|_{W^{r,s}_\omega} \). One of them is stated in the following theorem.

**Theorem 6.13.** Let (6.1) hold and \( r, s > \frac{1}{2} \). If \( v \in H_0^{r,\omega}(\Omega) \cap H_p^s(\Lambda_2, H_\omega^{s}(\Lambda_1)) \), then

\[
\| P_{M,N}^1 v \|_{L^\infty} \leq c \| v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))}.
\]

If in addition \( v \in X_{p,\omega}^{r,s}(\Omega) \), then

\[
\| P_{M,N}^1 v \|_{W^{1,\infty}} \leq c \| v \|_{X_{p,\omega}^{r,s}}.
\]

Finally, we consider the mixed Fourier-Chebyshev interpolation. Let \( I_M \) be the same as in the previous paragraphs and \( I_N \) be the Fourier interpolation on the interval \( \Lambda_2 \). Set \( I_{M,N} = I_M I_N = I_N I_M \).

**Theorem 6.14.** If \( v \in H_\omega^\beta(\Lambda_2, H_\omega^\alpha(\Lambda_1)) \cap H_0^\alpha(\Lambda_2, H_\omega^\alpha(\Lambda_1)) \cap H_\omega^\beta(\Lambda_2, H_\omega^\alpha(\Lambda_1)) \), \( 0 \leq \alpha \leq \min(r, \lambda), 0 \leq \beta \leq \min(s, \gamma) \) and \( r, s, \lambda, \gamma > \frac{1}{2} \), then

\[
\| v - I_{M,N} v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))} \leq \frac{c M^{2\alpha - r} \| v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))} + c N^{\beta - s} \| v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))}}{M^{2\alpha - \lambda} N^{\beta - \gamma} \| v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))}} + c q(\beta) M^{2\alpha - \lambda} N^{\beta - \gamma} \| v \|_{H^s(\Lambda_2, H_\omega^s(\Lambda_1))}.
\]

where \( q(\beta) = 0 \) for \( \beta > \frac{1}{2} \) and \( q(\beta) = 1 \) for \( \beta \leq \frac{1}{2} \).

The above theorems can be found in Guo, Ma, Cao and Huang (1992), Guo and Li (1995), and Guo (1998). Quarteroni (1987) also considered this kind of mixed approximation.

The filterings are also available for the mixed spectral methods. For example, let

\[
\phi(x) = \sum_{m=0}^{M} \sum_{|l|=0}^{N} \hat{\phi}_{m,l} T_m(x_1) e^{\theta x_2}.
\]

For \( \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 1 \), we define the filtering \( R_{M,N}(\alpha_1, \alpha_2, \beta_1, \beta_2) \) as

\[
R_{M,N}(\alpha_1, \alpha_2, \beta_1, \beta_2) \phi(x) = \sum_{m=0}^{M} \left( 1 - \frac{|m|}{M} \right)^{\alpha_1} \left( 1 - \frac{n}{N} \right)^{\alpha_2} \hat{\phi}_{m,l} T_m(x_1) e^{\theta x_2}.
\]

Following the same line as in the proofs of Theorem 2.26 and Theorem 2.27, we can prove that if \( 0 \leq r \leq \alpha_1 \) and \( 0 \leq s \leq \alpha_2 \), then

\[
\| R_{M,N}(\alpha_1, \alpha_2, \beta_1, \beta_2) \phi - \phi \|_{\omega} \leq c \left( \beta_1 M^{-r} + \beta_2 N^{-s} \right) \| \phi \|_{H_\omega^{r,s}}.
\]

In order to keep the spectral accuracy, we can take \( \alpha_j = \alpha_j(N) \) and \( \alpha_j(N) \to \infty \) as \( N \to \infty, j = 1,2 \).
6.3. Applications

We now take two-dimensional unsteady Navier-Stokes equation as an example to describe the mixed spectral methods. For simplicity, we focus on the mixed Fourier-Legendre spectral schemes. Let $\Lambda_1, \Lambda_2$ and $\Omega$ be the same as in the previous sections, and $\partial \Omega = \{ x \mid x_1 = -1, 1, 0 \leq x_2 \leq 2\pi \}$. $U(x, t), P(x, t)$ and $\mu \geq 0$ are the speed, the pressure and the kinetic viscosity, respectively, and $U = (U^{(1)}, U^{(2)})$. $U_0(x), P_0(x)$ and $f(x, t)$ are given functions with the period $2\pi$ for the variable $x_2$. We look for the solution $(U, P)$ with the period $2\pi$ for the variable $x_2$, such that

$$
\begin{align*}
\partial_t U + (U \cdot \nabla) U - \mu \Delta U + \nabla P &= f, & x \in \Omega, 0 < t < T; \\
\nabla \cdot U &= 0, & x \in \Omega, 0 \leq t \leq T; \\
U &= 0, & x \in \partial \Omega, 0 < t \leq T; \\
U(x, 0) &= U_0(x), & x \in \Omega, \\
P(x, 0) &= P_0(x), & x \in \Omega.
\end{align*}
$$

(6.2)

Let $$A(v) = \int_\Omega v(x) \, dx$$
and $L^2_{0,p}(\Omega) = \{ v \mid v \in L^2(\Omega), A(v) = 0 \}$. For fixing the values of $P$, we require that $P(x, t) \in L^2_{0,p}(\Omega)$ for all $0 \leq t \leq T$. Let $V_p = \{ v \mid v \in V_p(\Omega), \nabla \cdot v = 0 \}$. Denote by $H_p$ and $V_p$ the closures of $V_p$ in $L^2(\Omega)$ and $H^1(\Omega)$. Set

$$
a(v, w) = (\nabla v, \nabla w), \quad b(v, w) = (\nabla \cdot v, w).$$

(6.3)

The weak form of (6.2) is

$$
\begin{align*}
\{ (\partial_t U(t), v) + ((U(t) \cdot \nabla) U(t), v) + \mu a(U(t), v) &= (f, v), & \forall v \in V_p, 0 < t \leq T; \\
U(0) &= U_0, & x \in \Omega.
\end{align*}
$$

(6.4)

If $U_0 \in H_p$ and $f \in L^2(0, T; V'_p)$, then (6.4) possesses a unique solution in $L^2(0, T; V_p) \cap L^\infty(0, T; H_p)$. The solution satisfies the conservation which is exactly the same as (5.17). We can approximate (6.4) directly with trial function space belonging to the space $V_p$. To avoid this difficult job, we use artificial compression. It means that instead of the continuity equation, we consider the approximate one as

$$
\beta \partial_t P + \nabla \cdot U = 0, \quad \beta > 0.
$$

(6.5)

The solution of this auxiliary problem tends to the genuine solution of (6.4) as $\beta \to 0$, under some conditions. On the other hand, the convective term $U \cdot \nabla U$ can be expressed as $d(U, U)$ where

$$
d(v, w) = \frac{1}{2} w^{(1)}_1 \partial_1 v + \frac{1}{2} w^{(2)}_2 \partial_2 v + \frac{1}{2} \partial_1 \left( w^{(1)}_1 v \right) + \frac{1}{2} \partial_2 \left( w^{(2)}_2 v \right).
$$

(6.6)

Clearly

$$(d(v, w), z) + (d(z, w), v) = 0, \quad \forall v, z \in H^1(\Omega), w \in H^1_0(\Omega).$$

Let $M$ and $N$ be any positive integers, and $\tau$ be the mesh size in time. Set

$$
V_{M,N} = \mathbb{P}^{0}_{M,1} \otimes V_{N,2}, \quad W_{M,N} = \{ w \in \mathbb{P}^{0}_{M,1} \otimes V_{N,2} \mid A(v) = 0 \}.
$$
The approximations of $U$ and $P$ are denoted by $u_{M,N}$ and $p_{M,N}$, respectively. A mixed Fourier-Legendre spectral scheme with the artificial compression for (6.2) is to find $u_{M,N}(x,t) \in V_{M,N}$ and $p_{M,N}(x,t) \in W_{M,N}$ for all $t \in \bar{R}_r(T)$ such that

$$
\begin{align*}
\begin{cases}
(D_t u_{M,N}(t) + d(u_{M,N}(t) + \delta D_x u_{M,N}(t), u_{M,N}(t)), v) \\
+ \mu a(u_{M,N}(t) + \sigma D_x u_{M,N}(t), v) - b(v, p_{M,N}(t) + \theta D_x p_{M,N}(t)) = (f(t), v), \\
\forall \, v \in V_{M,N}, \, t \in \bar{R}_r(T), \\
\beta(D_t p_{M,N}(t), w) + b(u_{M,N}(t) + \theta D_x u_{M,N}(t), w) = 0, \, \forall \, w \in W_{M,N}, \, t \in \bar{R}_r(T), \\
u_{M,N}(0) = P^1_{M,N} u_0, \\
p_{M,N}(0) = P_{M,N} p_0,
\end{cases}
\end{align*}
$$

(6.7)

where $\delta, \sigma$ and $\theta$ are parameters and $0 \leq \delta, \sigma, \theta \leq 1$. If $\delta = \sigma = \theta = 0$, then (6.6) is an explicit scheme. Otherwise, we need an iteration to evaluate the values of $u_{M,N}$ and $p_{M,N}$ at each time step. In particular, if $\delta = \sigma = \theta = \frac{1}{2}$, then

$$
\begin{align*}
||u_{M,N}(t)||^2 + \beta ||p_{M,N}(t)||^2 + \frac{1}{2} \mu \tau \sum_{s \in \bar{R}_r(1-\tau)} |u_{M,N}(s) + u_{M,N}(s+\tau)|^2 \\
= ||u_{M,N}(0)||^2 + \beta ||p_{M,N}(0)||^2 + \tau \sum_{s \in \bar{R}_r(1-\tau)} (f(s), u_{M,N}(s) + u_{M,N}(s+\tau)).
\end{align*}
$$

This simulates the conservation (5.17) suitably.

We now analyse the generalized stability of (6.7). For simplicity, we only consider the case with $\delta = \sigma = 0, \theta > \frac{1}{2}$. Assume that $u_{M,N}(0), p_{M,N}(0), f$ and the right-hand side of the second formula of (6.7) have the errors $\tilde{u}, \tilde{p}, f$ and $\tilde{g}$ respectively. They induce the errors of $u_{M,N}$ and $p_{M,N}$, denoted by $\bar{u}$ and $\bar{p}$ respectively. Then we get from (6.7) that

$$
\begin{align*}
\begin{cases}
(D_t \bar{u}(t) + d(\bar{u}(t), u_{M,N}(t) + \bar{u}(t)) + d(u_{M,N}(t), \bar{u}(t)), v) + \mu a(\bar{u}(t), v) \\
- b(v, \bar{p}(t) + \theta D_x \bar{p}(t)) = (\tilde{f}, v), \, \forall \, v \in V_{M,N}, \, t \in \bar{R}_r(T), \\
\beta(D_t \bar{p}(t), w) + b(u_{M,N}(t) + \theta D_x \bar{u}(t), w) = (\tilde{g}, w), \, \forall \, w \in W_{M,N}, \, t \in \bar{R}_r(T), \\
\bar{u}(0) = \bar{u}_0, \, \bar{x} \in \bar{\Omega}, \\
\bar{p}(0) = \bar{p}_0, \, \bar{x} \in \bar{\Omega}.
\end{cases}
\end{align*}
$$

Assume that $\varepsilon$ and $q_0$ are suitably small positive constants such that

$$
2\theta - 1 - 3\varepsilon - q_0 \geq \theta \mu \tau \left(\frac{9}{4} M^4 + N^2\right) .
$$

(6.8)

Let

$$
\begin{align*}
E(v, w, t) &= ||v(t)||^2 + \beta ||w(t)||^2 + \tau \sum_{s \in \bar{R}_r(1-\tau)} (q_0 \tau ||D_x v(s)||^2 \\
&\quad + q_0 \beta \tau ||D_x w(s)||^2 + |v(s)|^2) \\
\rho(t) &= ||\bar{u}_0||^2 + \beta ||\bar{p}_0||^2 + \theta \mu \tau ||\bar{u}_0||^2 + \tau \sum_{s \in \bar{R}_r(1-\tau)} A_3(s). \varepsilon
\end{align*}
$$

Guo (1998) proved the following result.
Theorem 6.15. Let \( \delta = \sigma = 0, \theta > \frac{1}{2}, \tau \leq b_1 \mu \) in (6.7), and (6.8) hold. If
\[
\rho(t) e^{b_2 t} \leq \frac{b_1}{T^2 N^2} \text{ for some } t_1 \in R_r(T), \text{ then for all } t \in \bar{R}_r(t_1),
\]

\[E(\tilde{u}, \tilde{p}, t) \leq \rho(t) e^{b_2 t}\]

where \( b_1, b_2 \) and \( b_3 \) are some positive constants depending on \( \mu \) and \( ||u_{M,N}||_{C([0,T]; W^{1,\infty})} \).

We next deal with the convergence of (6.7) with \( \delta = \sigma = 0 \) and \( \theta > \frac{1}{2} \). Let
\[
U_{M,N} = P_{M,N}^1 U, \bar{U} = u_{M,N} - U_{M,N} \text{ and } \bar{P} = p_{M,N} - P_{M,N} P. \text{ We get from (6.2) and (6.7) that}
\]

\[
\begin{aligned}
\begin{cases}
D_r \bar{U}(t) + d \left( \bar{U}(t), U_{M,N}(t) + \bar{U}(t) \right) + d \left( U_{M,N}(t), \bar{U}(t) \right), v \right) + \mu a \left( \bar{U}(t), v \right) \\
- b \left( v, \bar{P}(t) + \theta \tau D_r \bar{P}(t) \right) = (G_1(t) + G_2(t) + G_3(t), v) - b \left( v, G_4(t) \right), \\
\beta \left( D_r \bar{P}(t), w \right) + b \left( \bar{U}(t) + \theta \tau D_r \bar{U}(t), w \right) = - \beta \left( D_r \bar{P}(t), w \right) + b \left( G_5(t), w \right),
\end{cases}
\forall \ v \in V_{M,N}, \ t \in \bar{R}_r(T),
\forall \ w \in W_{M,N}, \ t \in \bar{R}_r(T),
\end{aligned}
\]

where
\[
G_1(t) = \partial_t U(t) - D_r U_{M,N}(t),
G_2(t) = (U(t) \cdot \nabla) U(t) - (U_{M,N}(t) \cdot \nabla) U_{M,N}(t),
G_3(t) = - \frac{1}{2} \left( \nabla \cdot U_{M,N}(t) \right) U_{M,N}(t),
G_4(t) = P(t) - P_{M,N} P(t) - \theta \tau D_r P_{M,N} P(t),
G_5(t) = \nabla \cdot (U(t) - U_{M,N}(t) - \theta \tau D_r U_{M,N}(t)).
\]

Guo (1996) proved the following result.

Theorem 6.16. Let \( \delta = \sigma = 0, \theta > \frac{1}{2}, \tau \leq b_4 \mu \) in (6.7), and (6.8) hold. If for \( r, s \geq 1 \),
\[
U \in C \left( (0,T; H^{1}_{0,p} \cap M^{r+1,s+1} \cap W^{1,\infty}) \cap H^1(0,T; M^{r,s}_p) \cap H^2(0,T; L^2_p) \right),
P \in C \left( (0,T; H^r_p) \cap H^1(0,T; L^2_p) \right),
\]

then for all \( t \in \bar{R}_r(T) \),

\[E(\bar{u}_{M,N} - \bar{U}_{M,N}, \bar{p}_{M,N} - \bar{P}_{M,N} P, t) \leq b_5 \left( 1 + a \right) \beta \left( \tau^2 + M^{-2r} + N^{-2s} \right) + \beta \]

where \( b_4 \) and \( b_5 \) are some positive constants depending on \( \mu \) and the norms of \( \bar{U} \) and \( \bar{P} \) in the spaces mentioned above.

We find from Theorem 6.16 that the best choice is \( \beta = O(\tau) = O \left( M^{-2r} + N^{-2s} \right) \),
and the corresponding accuracy is of the order \( \beta^2 \). It seems that the artificial compression brings the convenience in calculations, but lowers the accuracy. Indeed, if
we compare the numerical solution to other projection of the exact solution, called the Stokes projection, we can remove this trouble, and get a better error estimation for any $\beta \geq 0$.

Obviously, since we use the approximation (6.6) for the nonlinear term $(U \cdot \nabla) U$, a reasonable simulation of conservation follows, even the trial functions do not satisfy the incompressibility. Also because of this reason, the effects of the main nonlinear error terms are cancelled, i.e.,

$$(d(\bar{u}(t), \bar{u}(t)), \bar{u}(t)) = \left(d\left(\bar{U}(t), \bar{U}(t)\right), \bar{U}(t)\right) = 0.$$  

Therefore the nice error estimates follow. Conversely, if the trial function space $V_{M,N}$ fulfills the incompressibility, then it is better to approximate the convective term by $\tilde{d}(u_{M,N}, u_{M,N})$,

$$\tilde{d}(v, w) = \partial_t \left( w^{(1)} v \right) + \partial_t \left( w^{(2)} v \right).$$  

In this case, the conservation can also be simulated properly. For the error analysis, we have

$$\left(\tilde{d}(\bar{u}(t), \bar{u}(t)), \bar{u}(t)\right) = \frac{1}{2} \left( |\bar{u}(t)|^2, \nabla \cdot \bar{u}(t) \right).$$  

Since $\nabla \cdot \bar{u}(t) \neq 0$ usually, this term cannot be cancelled. This fact shows that a good approximation to the nonlinear convective term plays an important role.

Bernardi, Maday and Metivet (1987) applied the mixed Fourier-Legendre spectral methods first to semi-periodic problems of steady Navier-Stokes equation.

The mixed Fourier-Chebyshev spectral methods have been used also for semi-periodic problems of various nonlinear differential equations, such as the vorticity equation in Guo, Ma, Cao and Huang (1992), and the Navier-Stokes equation in Guo and Li (1996). Recently, the Fourier-Chebyshev pseudospectral methods have also been developed, e.g., see Guo and Li (1993, 1995).
Lecture 7

Combined Spectral Methods

In many practical problems, the domains are not rectangular. However, sometimes the intersections of the domains may be rectangular, such as a cylindrical container. So it is reasonable to use the combined spectral-finite element methods or the combined pseudospectral-finite element methods. Since finite element methods are based on Galerkin method with interpolations of functions, the latter seem more attractive.

7.1. Some Basic Results In Finite Element Methods

We first recall some basic results of finite element methods in one dimension, as a preparation of the forthcoming discussions. Let $\Lambda = (-1, 1)$ and $x \in \Lambda$. Decompose $\Lambda$ into $2M$ subintervals $\Delta_j, -M \leq j \leq M - 1$. Let $h = \max_{-M \leq j \leq M - 1} h_j$. Assume that there exists a positive constant $\rho$ independent of the partition such that $\max_{-M \leq j \leq M - 1} \frac{h}{h_j} \leq \rho$. Let $k$ be non-negative integer and $\mathcal{S}_{h,k}$ be the Lagrange interpolation of degree $k$ over each $\Delta_j$. Define

$$\mathcal{S}_{h,k} = \{ v \mid v \text{ is a polynomial of degree } \leq k \text{ on } \Delta_j, -M \leq j \leq M - 1 \}.$$ 

Moreover $\mathcal{S}_{h,k} = \mathcal{S}_{h,k} \cap H^1(\Lambda)$ and $\mathcal{S}_{h,k}^0 = \mathcal{S}_{h,k} \cap H^1_0(\Lambda)$. In the space $\mathcal{S}_{h,k}$, several inverse inequalities exist.

**Lemma 7.1.** For any $\phi \in \mathcal{S}_{h,k}$ and $1 \leq p \leq q \leq \infty$,

$$\|\phi\|_{L^q} \leq c \left( \frac{h}{k''} \right)^{\frac{1}{2} - \frac{1}{q}} \|\phi\|_{L^p}.$$ 

**Lemma 7.2.** For any $\phi \in \mathcal{S}_{h,k}$ and $0 \leq r \leq \mu$,

$$\|\phi\|_{r} \leq c \left( \frac{h}{k''} \right)^{r - \mu} \|\phi\|_{r}.$$ 

The $L^2(\Lambda)$-orthogonal projection $P_{h,k} : L^2(\Lambda) \rightarrow \mathcal{S}_{h,k}$ is such a mapping that for any $v \in L^2(\Lambda)$,

$$(v - P_{h,k}v, \phi) = 0, \quad \forall \phi \in \mathcal{S}_{h,k}. \quad (7.1)$$
For simplicity, let \( c(k) \) be a generic positive constant depending on \( k \). We have from Ciarlet (1978) that for any \( v \in H^r(\Lambda), \frac{1}{2} < r \leq k + 1 \) and \( 0 \leq \mu \leq \min(r, 1) \),
\[
\|v - \Gamma_{h,k} v\|_{\mu} \leq c(k) h^{r-\mu} |v|_{r},
\]
\[
\|v - \Gamma_{h,k} v\|_{W^{\mu,-}} \leq c(k) h^{r-\mu} |v|_{W^{r,-}}.
\]

**Lemma 7.3.** For any \( v \in H^r(\Lambda), \frac{1}{2} < r \leq k + 1 \) and \( 0 \leq \mu \leq \min(r, 1) \),
\[
\|v - P_{h,k} v\|_{\mu} \leq c(k) h^{r-\mu} \|v\|_{r}.
\]

The \( H^1(\Lambda) \)-orthogonal projection: \( H^1(\Lambda) \to S_{h,k} \) is such a mapping that for any \( v \in H^1(\Lambda) \),
\[
(\partial_x (v - P_{h,k}^1 v), \partial_x \phi) + (v, \phi) = 0, \quad \forall \phi \in S_{h,k}.
\]

The \( H^0_0(\Lambda) \)-orthogonal projection: \( H^0_0(\Lambda) \to S^0_{h,k} \) is such a mapping that for any \( v \in H^0_0(\Lambda) \),
\[
(\partial_x (v - P_{h,k}^0 v), \partial_x \phi) = 0, \quad \forall \phi \in S^0_{h,k}.
\]

**Lemma 7.4.** If \( v \in H^r(\Lambda), 1 \leq r \leq k + 1 \) and \( 1 - k \leq \mu \leq 1 \), then
\[
\|v - P_{h,k}^1 v\|_{\mu} \leq c(k) h^{r-\mu} \|v\|_{r}.
\]

**Lemma 7.5.** If \( v \in H^r(\Lambda) \cap H^0_0(\Lambda), 1 \leq r \leq k + 1 \) and \( 1 - k \leq \mu \leq 1 \), then
\[
\|v - P_{h,k}^0 v\|_{\mu} \leq c(k) h^{r-\mu} |v|_{r}.
\]

Lemma 7.5 comes from Douglas and Dupont (1974). Lemma 7.4 can be proved in the same manner. Guo and Ma (1991) gave a result similar to Lemma 7.3.

### 7.2. Combined Fourier-Finite Element Approximations

We discuss the combined Fourier-finite element approximations in this part. Let \( \Lambda_1 = \{x_1 \mid |x_1| < 1\}, \Lambda_2 = \{x_2 \mid 0 < x_2 < 2\pi\} \) and \( \Omega = \Lambda_1 \times \Lambda_2 \). We use the notations in Section 6.1, such as \( H^r_p(\Omega), H^0_p(\Omega), M^r_p(\Omega) \) and \( X^r_p(\Omega) \). \( k \) and \( N \) are integers, \( k \geq 0, N > 0 \). \( \tilde{S}_{h,k,1} \) and \( S_{h,k,1} \) are the sets of functions on \( \Lambda_1 \), as defined in the above section. \( S^0_{h,k,1} = \tilde{S}_{h,k,1} \cap H^0_0(\Lambda_1) \). \( V_{N,2} \) is the set of trigonometric functions of degree \( \leq N \), on \( \Lambda_2 \). Set \( \tilde{V}_{h,k,N} = \tilde{S}_{h,k,1} \cap V_{N,2}, \tilde{V}_{h,k,N} = S_{h,k,1} \cap V_{N,2} \) and \( V^0_{h,k,N} = S^0_{h,k,1} \cap V_{N,2} \). Several inverse inequalities are valid in the space \( \tilde{V}_{h,k,N} \).

**Theorem 7.1.** For any \( \phi \in \tilde{V}_{h,k,N} \) and \( 2 \leq p \leq \infty \),
\[
\|\phi\|_{L^p} \leq c \left( \frac{h}{k^2 N} \right)^{\frac{1}{p} - \frac{1}{r}} \|\phi\|.
\]

**Theorem 7.2.** For any \( \phi \in \tilde{V}_{h,k,N} \) and \( 0 \leq r \leq \mu \),
\[
\|\phi\|_{\mu} \leq c \left( \frac{h}{k^2 + \frac{1}{N}} \right)^{r-\mu} \|\phi\|_{r}.
\]
The $L^2(\Omega)$-orthogonal projection $P_{h,k,N} : L^2_p(\Omega) \to \overline{V}_{h,k,N}$ is such a mapping that for any $v \in L^2_p(\Omega)$,

$$(v - P_{h,k,N} v, \phi) = 0, \quad \forall \phi \in \overline{V}_{h,k,N}.$$ 

Set $\overline{r} = \min(r, k + 1)$. For simplicity, $P_{h,k}$ denotes the $L^2(\Lambda_1)$-orthogonal projection on $S_{h,k,1}$, and $P_N$ denotes the $L^2(\Lambda_2)$-orthogonal projection on $V_{N,2}$. Similarly, $P_{h,k}^1$ stands for the $H^1(\Lambda_1)$-orthogonal projection on $S_{h,k,1}$, and $P_{h,k}^{1,0}$ stands for the $H^1_0(\Lambda_1)$-orthogonal projection on $S_{h,k,1}^0$.

**Theorem 7.3.** If $v \in H^r_p(\Omega), r > \frac{1}{2}$ and $s \geq 0$, then

$$\|v - P_{h,k,N} v\| \leq c(k) \left( h^{\overline{r}} + N^{-s} \right) \|v\|_{H^r,\sigma}.$$ 

In particular, for any $v \in H^r_{0,p}(\Omega), r > \frac{1}{2}$ and $s \geq 0$,

$$\|v - P_{h,k,N} v\| \leq c(k) \left( h^{\overline{r}} + N^{-s} \right) \|v\|_{H^r,\sigma}.$$ 

The $H^1(\Omega)$-orthogonal projection $P_{h,k,N}^1 : H^1_p(\Omega) \to \overline{V}_{h,k,N}$ is such a mapping that for any $v \in H^1_p(\Omega)$,

$$\left( \nabla \left( v - P_{h,k,N}^1 v \right), \nabla \phi \right) + (v, \phi) = 0, \quad \forall \phi \in \overline{V}_{h,k,N}.$$ 

The $H^1_0(\Omega)$-orthogonal projection $P_{h,k,N}^{1,0} : H^1_{0,p}(\Omega) \to \overline{V}_{h,k,N}^0$ is such a mapping that for any $v \in H^1_{0,p}(\Omega)$,

$$\left( \nabla \left( v - P_{h,k,N}^{1,0} v \right), \nabla \phi \right) = 0, \quad \forall \phi \in \overline{V}_{h,k,N}^0.$$ 

**Theorem 7.4.** If $v \in M^r_{0,p}(\Omega)$ and $r, s \geq 1$, then

$$\|v - P_{h,k,N} v\| \leq c(k) \left( h^{\overline{r}-1} + N^{1-s} \right) \left( h^{1-\mu} + N^{\mu - 1} \right) \|v\|_{M^r,\sigma}, \quad \mu = 0, 1.$$ 

If, in addition, for certain positive constants $c_1$ and $c_2$,

$$c_1 N \leq \frac{1}{h} \leq c_2 N,$$ 

then

$$\|v - P_{h,k,N}^1 v\| \leq c(k) \left( h^{\overline{r}} + N^{-s} \right) \|v\|_{M^r,\sigma}.$$ 

**Theorem 7.5.** If $v \in M^r_{0,p}(\Omega) \cap H^1_{0,p}(\Omega)$ and $r, s \geq 1$, then

$$\|v - P_{h,k,N}^{1,0} v\| \leq c(k) \left( h^{\overline{r}-1} + N^{1-s} \right) \left( h^{1-\mu} + N^{\mu - 1} \right) \|v\|_{M^r,\sigma}, \quad \mu = 0, 1.$$ 

If in addition (7.4) holds, then

$$\|v - P_{h,k,N}^{1,0} v\| \leq c(k) \left( h^{\overline{r}} + N^{-s} \right) \|v\|_{M^r,\sigma}.$$ 

We can also bound the norm $\|P_{h,k,N} v\|_{W^{r,p}}$. 

Theorem 7.6. Let (7.4) hold, $k \geq 1$ and $r > \frac{1}{2}$. For any $v \in H^1_{0,\beta}(\Omega) \cap H^r_p(\lambda_2, H^r(\lambda_1)),$

$$
\left\| P_{h,k,N}^1 v \right\|_{L^\infty} \leq c(k) \|v\|_{H^r(\lambda_2, H^r(\lambda_1))}.
$$

If in addition $v \in X^p_\sigma(\Omega)$, then

$$
\left\| P_{h,k,N}^1 v \right\|_{W^{1,\infty}} \leq c(k) \|v\|_{X^p_\sigma}.
$$

Finally we consider the combined Fourier pseudospectral-finite element approximation. Let $I_N$ be the Fourier interpolation on the interval $\lambda_2$, associated with $\lambda_{2,N}$, the set of interpolation points. Define $I_{h,k,N} = P_{h,k}I_N = I_N P_{h,k}$. Set

$$(v, w)_N = \frac{2\pi}{2N+1} \sum_{x_k \in \Lambda_{2,N}} \int_{\Lambda_1} v(x) w(x) \, dx, \quad \|v\|_N = (v, v)_N^{1/2}.$$

Then for any $v \in C(\lambda_2, L^2(\lambda_1))$,

$$(v - I_{h,k,N} v, \phi)_N = 0, \quad \forall \phi \in \tilde{V}_{h,k,N}.$$

Theorem 7.7. If $v \in H^r_p(\Omega)$ and $r, s > \frac{1}{2}$, then

$$
\|v - I_{h,k,N} v\| \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{H^r_p}.
$$


7.3. Combined Legendre-Finite Element Approximations

We now consider non-periodic problems with the intersection of the domains rectangular. In this case, we can use the combined Legendre-finite element approximations. For sake of simplicity, we only consider two-dimensional problems. Clearly it is not necessary to use such kind of combined approximations for two-dimensional problems. But it is easy to generalize the corresponding results to the three-dimensional problems.

Let $\lambda_1 = \{x_1 \mid |x_1| < 1\}$, $\lambda_2 = \{x_2 \mid |x_2| < 1\}$ and $\Omega = \lambda_1 \times \lambda_2$. The integers $k \geq 0$ and $N > 0$. The notations $h, \tilde{S}_{h,k,1}, S_{h,k,1}, S^0_{h,k,1}, P_{h,k}, P^1_{h,k}$ and $P^1_{h,k}$ have the same meanings as the previous part. Let $B_{N,2}$ be the set of all monomials of degree at most $N$ defined on $\lambda_2$, and $B^0_{N,2} = P^0_{N,2} \cap H^0_0(\lambda_2)$. For simplicity, let $P_N, P^0_N$ and $P^1_N$ denote the $L^2(\lambda_2)$-orthogonal projection on $P_{N,2}$, the $H^1(\lambda_2)$-orthogonal projection on $P_{N,2}$ and the $H^1_0(\lambda_2)$-orthogonal projection on $P^0_{N,2}$, respectively. Set $\tilde{V}_{h,k,N} = \tilde{S}_{h,k,1} \otimes P_{N,2}, V_{h,k,N} = S_{h,k,1} \otimes P_{N,2}$ and $V^0_{h,k,N} = V_{h,k,N} \cap H^0_0(\Omega)$. In the space $\tilde{V}_{h,k,N}$, several inequalities are valid.

Theorem 7.8. For any $\phi \in \tilde{V}_{h,k,N}$ and $2 \leq p \leq \infty$,

$$
\| \phi \|_{L^p} \leq c \left( \frac{h}{k^2 N^2} \right)^{\frac{1}{p-1}} \| \phi \|.
$$
Theorem 7.9. For any \( \phi \in \tilde{V}_{h,k,N} \) and \( 0 \leq r \leq \mu \),

\[
\|\phi\|_{\mu} \leq c \left( \frac{h}{N^2} + \frac{1}{N^2} \right)^{r-\mu} \|\phi\|_r.
\]

The \( L^2(\Omega) \)-orthogonal projection \( P_{h,k,N} : L^2(\Omega) \rightarrow \tilde{V}_{h,k,N} \) is such a mapping that for any \( v \in L^2(\Omega) \),

\[
(v - P_{h,k,N} v, \phi) = 0, \quad \forall \phi \in \tilde{V}_{h,k,N}.
\]

Theorem 7.10. If \( v \in H^{r,s}(\Omega), r > \frac{1}{2} \) and \( s \geq 0 \), then

\[
\|v - P_{h,k,N} v\| \leq c(k) \left( \frac{h}{N} + N^{-s} \right) \|v\|_{H^{r,s}}.
\]

In particular, for \( v \in H^{r,s}_0(\Omega) \),

\[
\|v - P_{h,k,N} v\| \leq c(k) \left( \frac{h}{N} + N^{-s} \right) |v|_{H^{r,s}}.
\]

The \( H^1(\Omega) \)-orthogonal projection \( P_{h,k,N}^1 : H^1(\Omega) \rightarrow \tilde{V}_{h,k,N} \) is such a mapping that for any \( v \in H^1(\Omega) \),

\[
\left( \nabla \left( v - P_{h,k,N}^1 v \right), \nabla \phi \right) + (v, \phi) = 0, \quad \forall \phi \in \tilde{V}_{h,k,N}.
\]

The \( H^1_0(\Omega) \)-orthogonal projection \( P_{h,k,N}^{1,0} : H^1_0(\Omega) \rightarrow V_{h,k,N}^0 \) is such a mapping that for any \( v \in H^1_0(\Omega) \),

\[
\left( \nabla \left( v - P_{h,k,N}^{1,0} v \right), \nabla \phi \right) = 0, \quad \forall \phi \in V_{h,k,N}^0.
\]

Theorem 7.11. If \( v \in M^{r,s}(\Omega) \) and \( r, s \geq 1 \), then for \( \mu = 0, 1 \),

\[
\left\| v - P_{h,k,N}^1 v \right\|_\mu \leq c(k) \left( \frac{h}{N} + h^{-1} N^{-s} + N^{1-s} \right) \left( h^{1-\mu} + h \right) \|v\|_{M^{r,s}}.
\]

If, in addition, (7.4) holds, then

\[
\left\| v - P_{h,k,N}^1 v \right\| \leq c(k) \left( \frac{h}{N} + N^{-s} \right) |v|_{M^{r,s}}.
\]

Theorem 7.12. If \( v \in M^{r,s}(\Omega) \cap H^1_0(\Omega) \) and \( r, s \geq 1 \), then for \( \mu = 0, 1 \),

\[
\left\| v - P_{h,k,N}^{1,0} v \right\|_\mu \leq c(k) \left( \frac{h}{N} + h^{-1} N^{-s} + N^{1-s} \right) \left( h^{1-\mu} + h \right) \|v\|_{M^{r,s}}.
\]

If, in addition, (7.4) holds, then

\[
\left\| v - P_{h,k,N}^{1,0} v \right\| \leq c(k) \left( \frac{h}{N} + N^{-s} \right) |v|_{M^{r,s}}.
\]

We now estimate \( \left\| P_{h,k,N}^1 v \right\|_{W^{-r,s}} \).

Theorem 7.13. Let (7.4) hold, \( k \geq 1 \) and \( r > \frac{1}{2} \). For any \( v \in M^{\frac{3}{2},\frac{3}{2}}(\Omega) \cap H^1_0(\Omega) \cap H^{\frac{3}{2}}(\Lambda_2, H^r(\Lambda_1)) \),

\[
\left\| P_{h,k,N}^1 v \right\|_{L^\infty} \leq c(k) \|v\|_{M^{\frac{3}{2},\frac{3}{2}}(\Omega) \cap H^1_0(\Omega) \cap H^{\frac{3}{2}}(\Lambda_2, H^r(\Lambda_1))}.
\]

If, in addition, \( k \geq 2 \) and \( v \in M^{\frac{3}{2},\frac{3}{2}}(\Omega) \cap X^{\frac{1}{2},\frac{3}{2}}(\Omega) \cap X^{r,\frac{3}{2}}(\Omega) \), then

\[
\left\| P_{h,k,N}^1 v \right\|_{W^{-r,s}} \leq c(k) \|v\|_{M^{\frac{3}{2},\frac{3}{2}} \cap X^{\frac{1}{2},\frac{3}{2}} \cap X^{r,\frac{3}{2}}}.
\]
We now deal with the combined Legendre pseudospectral-finite element approximation. Let $I_N$ be the Legendre-Gauss-Lobatto interpolation on the interval $\Lambda_2$, associated with the interpolation points $x_2^{(j)}$ and the weights $\omega^{(j)}, 0 \leq j \leq N$. Define $I_{h,k,N} = P_{h,k} I_N = I_N P_{h,k}$. Set

$$(v, w)_N = \sum_{j=0}^N \int_{\Lambda_1} v(x_1, x_2^{(j)}) w(x_1, x_2^{(j)}) \omega^{(j)} \, dx_1, \quad \|v\|_N = (v, v)_N^\frac{1}{2}.$$ 

Then for any $v \in C(\Lambda_2, L^2(\Lambda_1))$,

$$(v - I_{h,k,N} v, \phi)_N = 0, \quad \forall \phi \in \bar{V}_{h,k,N}.$$ 

**Theorem 7.14.** If $v \in H^r_s(\Omega)$ and $r, s > \frac{1}{2}$, then

$$\|v - I_{h,k,N} v\| \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{H^r_s}.$$ 

The proof of the above theorems are given in Guo (1998).

**7.4. Combined Chebyshev-Finite Element Approximations**

This part is devoted to the combined Chebyshev-finite element approximations. Let $\Lambda_1, \Lambda_2$ and $\Omega$ be the same as in the previous part. Also, the notations $\bar{S}_{h,k,1}, \bar{S}_{h,k,1}, S_{h,k,1}, \bar{S}_{h,k,1}^0, \bar{P}_{N,2}, P_{N,2}, \bar{V}_{h,k,N}, \bar{V}_{h,k,N}, V_{h,k,N}, V_0, V_{h,k,N}, V_0$ and $P_{h,k}, P_{h,k}^1, P_{h,k}^1, P_{h,k}^1$ have the same meanings as in that section. Let $\omega(x) = \omega(x_2) = (1 - x_2^2)^{-\frac{1}{2}}$. Define

$$(v, w)_{\omega} = \int_{\Omega} v(x) w(x) \omega(x_2) \, dx, \quad \|v\|_{\omega} = (v, v)^{\frac{1}{2}}_{\omega}.$$ 

We still use the notations $L^p_0(\Omega), H^r_s(\Omega), H^r_s(\Omega), W^{r,p}_0(\Omega), H^r_s(\Omega), H^r_s(\Omega), M^r_s(\Omega)$ and $X^r_s(\Omega)$ as before. Moreover, $P_N, P_N^1$ and $P_N^{1,0}$ stand for the $L^2_0(\Lambda_2)$-orthogonal projection on $\bar{P}_{N,2}$, the $H^1_0(\Lambda_2)$-orthogonal projection on $\bar{P}_{N,2}$ and the $H^1_0(\Lambda_2)$-orthogonal projection on $\bar{P}_{N,2}$ respectively. We first give some inverse inequalities.

**Theorem 7.15.** For any $\phi \in \bar{V}_{h,k,N}$ and $2 \leq p \leq \infty$,

$$\|\phi\|_{L^p_0} \leq c \left( \frac{h}{k^2 N} \right)^{\frac{1}{2} - \frac{1}{p}} \|\phi\|_{\omega}.$$ 

**Theorem 7.16.** For any $\phi \in \bar{V}_{h,k,N}$ and $0 \leq r \leq \mu$,

$$\|\phi\|_{\mu, \omega} \leq c \left( \frac{h}{k^2} + \frac{1}{N^2} \right)^{\frac{1}{r} - \frac{1}{\mu}} \|\phi\|_{r, \omega}.$$ 

The $L^2_0(\Omega)$-orthogonal projection $P_{h,k,N} : L^2_0(\Omega) \to \bar{V}_{h,k,N}$ is such a mapping that for any $v \in L^2_0(\Omega)$,

$$(v - P_{h,k,N} v, \phi)_{\omega} = 0, \quad \forall \phi \in \bar{V}_{h,k,N}.$$
Theorem 7.17. If $v \in H^r_\omega(\Omega), r > \frac{1}{2}$ and $s \geq 0$, then

$$\|v - P_{h,k,N} v\|_\omega \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{H^r_\omega}.$$ 

In particular, for $v \in H^r_0(\omega)$,

$$\|v - P_{h,k,N} v\|_\omega \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{H^r_0}.$$ 

The $H^1_\omega(\Omega)$-orthogonal projection $P_{h,k,N}^1 : H^1_\omega(\Omega) \to V_{h,k,N}$ is such a mapping that for any $v \in H^1_\omega(\Omega)$,

$$\left( \nabla (v - P_{h,k,N}^1 v), \nabla \phi \right)_\omega + (v, \phi)_\omega = 0, \quad \forall \phi \in V_{h,k,N}.$$ 

The $H^1_0(\omega)$-orthogonal projection $P_{h,k,N}^{1,0} : H^1_0(\omega) \to V_{h,k,N}^0$ is such a mapping that for any $v \in H^1_0(\omega)$,

$$\left( \nabla (v - P_{h,k,N}^{1,0} v), \nabla (\phi) \right)_\omega = 0, \quad \forall \phi \in V_{h,k,N}^0.$$ 

Theorem 7.18. If $v \in M^r_\omega(\Omega)$ and $r, s \geq 1$, then

$$\|v - P_{h,k,N}^1 v\|_{\mu,\omega} \leq c(k) \left( h^{r-1} + h^{-1} N^{-s} + N^{1-s} \right) \left( h^{1-\mu} + N^{\mu-1} \right) \|v\|_{M^\mu_\omega}, \quad \mu = 0, 1.$$ 

If, in addition, (7.4) holds, then

$$\|v - P_{h,k,N}^1 v\|_\omega \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{M^\mu_\omega}.$$ 

Theorem 7.19. If $v \in M^r_\omega(\Omega) \cap H^1_0(\omega)$ and $r, s \geq 1$, then for $\mu = 0, 1$,

$$\|v - P_{h,k,N}^{1,0} v\|_{\mu,\omega} \leq c(k) \left( h^{r-1} + h^{-1} N^{-s} + N^{1-s} \right) \left( h^{1-\mu} + N^{\mu-1} \right) \|v\|_{M^\mu_\omega}.$$ 

If, in addition, (7.4) holds, then

$$\|v - P_{h,k,N}^{1,0} v\|_\omega \leq c(k) \left( h^r + N^{-s} \right) \|v\|_{M^\mu_\omega}.$$ 

We now estimate the norm $\|P_{h,k,N}^{1,0} v\|_{W^{r,p}_\omega}.$

Theorem 7.20. Let (7.4) hold, $k \geq 1$ and $r, s > \frac{1}{2}$. For any $v \in H^1_0(\Omega) \cap H^s_\omega(\Omega \setminus \Lambda_2, H^r(\Lambda_1))$,

$$\|P_{h,k,N}^{1,0} v\|_{L^\infty} \leq c(k) \|v\|_{H^1_0 \cap H^s_\omega(\Omega \setminus \Lambda_2, H^r(\Lambda_1))}.$$ 

If, in addition, $v \in M_{\omega}^{2,2}(\Omega) \cap X^r_\omega(\Omega)$, then

$$\|P_{h,k,N}^{1,0} v\|_{W^{1,\infty}} \leq c(k) \|v\|_{M_{\omega}^{2,2} \cap X^r_\omega}.$$
Finally we consider the combined Chebyshev pseudospectral-finite element approximation. Let \( I_N \) be the Chebyshev-Gauss-Lobatto interpolation on the interval \( \Lambda_2 \), associated with the interpolation points \( x_2^{(j)} \) and the weights \( \omega^{(j)} \), \( 0 \leq j \leq N \). Define \( I_{h,k,N} = P_{h,k}I_N = I_NP_{h,k} \). Set

\[
(v, w)_N = \sum_{j=0}^{N} \int_{\Lambda_1} v \left( x_1, x_2^{(j)} \right) w \left( x_1, x_2^{(j)} \right) \omega^{(j)} \, dx_1, \quad \| v \|_N = (v, v)_N^{\frac{1}{2}}.
\]

Then for any \( v \in C \left( \Lambda_2, L^2(\Lambda_1) \right) \),

\[
(v - I_{h,k,N}v, \phi)_N = 0, \quad \forall \, \phi \in \tilde{V}_{h,k,N}.
\]

**Theorem 7.21.** If \( v \in H_0^{r,s}(\Omega) \) and \( r > \frac{1}{2} \), then

\[
\| v - I_{h,k,N}v \|_\omega \leq c(k) \left( h^r + N^{-s} \right) \| v \|_{H_0^{r,s}}.
\]

Theorem 7.17, Theorem 7.19 and Theorem 7.20 can be found in Guo, He and Ma (1996). The others are due to Guo (1998).

### 7.5. Applications

We take the two-dimensional unsteady Navier-Stokes equation as an example again to describe the combined spectral-finite element methods. We focus on the combined Fourier pseudospectral-finite element schemes. But the main idea and the technique used can be generalized to non-periodic problems defined on a three-dimensional complex domain, and to other kinds of combined spectral-finite element methods and combined pseudospectral-finite element methods presented in the previous parts.

Let \( U, P \) and \( \mu > 0 \) be the speed vector, the pressure and the kinetic viscosity. We consider the semi-periodic problem (6.4) and require \( P(x, t) \) to be in the space \( L_0^{2,2}(\Omega) \), i.e., \( P \) has the period \( 2\pi \) for \( x_2 \) and \( A(P(x, t)) = 0 \) for \( 0 \leq t \leq T \). The bilinear forms \( a(v, w) \) and \( b(v, w) \) are given by (6.3). We adopt the artificial compression (6.5) with the parameter \( \beta \geq 0 \). When \( \beta = 0 \), it is reduced to the original problem and so requires the incompressibility automatically. A reasonable approximation of non-linear convective term plays an important role in preserving the global property of the genuine solution and in increasing the computational stability and the total accuracy of numerical solution. But it is not easy to do so for pseudospectral methods, since the interpolation and the differentiation do not commute. Besides for the improvement of stability, a filtering will be used.

Now let \( k \geq 0 \) and \( N > 0 \) be integers. We use all the notations of part 7.2. For simplicity, let \( V_{h,N} \) be the set of vector functions and \( W_{h,N} \) be the set of scalar functions, defined by

\[
V_{h,N} = \left( S_{h,k+1,1}^0 \otimes V_{N,2} \right)^2, \quad W_{h,N} = \left\{ w \mid w \in \tilde{S}_{h,k,1} \otimes V_{N,2}, A(w) = 0 \right\}.
\]

The interpolation \( \Pi_{h,k+1} \) is denoted by \( \Pi_{k+1} \) for simplicity. Furthermore for

\[
v(x) = \sum_{l=0}^{N} v_l(x_1) e^{ilx_2},
\]
the filtering with single parameter $\alpha \geq 1$ is given by

$$R_{NV} = R_N(\alpha) v = \sum_{i=0}^{N} \left( 1 - \left| \frac{I}{N} \right|^\alpha \right) v_i(x) e^{i\pi z}.$$ 

Let

$$D(v, w, z) = \frac{1}{2}((w \cdot \nabla)v, z) - \frac{1}{2}(w \cdot \nabla z, v).$$

If $\nabla \cdot w = 0$, then $D(v, w, z) = ((w \cdot \nabla)v, z)$. Thus we define the following trilinear form

$$d(v, w, z) = \frac{1}{2} (I_N((w \cdot \nabla)v), z) - \frac{1}{2} (I_N((w \cdot \nabla)z), v).$$

We shall approximate the nonlinear term by $d(v, v, z)$. Clearly

$$d(v, w, z) + d(z, w, v) = 0. \quad (7.5)$$

Guo and Ma (1993) proposed a scheme for (6.4). Let $\tau$ be the mesh size in time. $u_{h,N}$ and $p_{h,N}$ are the approximations to $U$ and $P$, respectively. $\delta, \sigma$ and $\theta$ are parameters, $0 \leq \delta, \sigma, \theta \leq 1$. The combined Fourier pseudospectral-finite element scheme for (6.4) is to find $u_{h,N}(x, t) \in V_{h,N}, p_{h,N}(x, t) \in W_{h,N}$ for all $t \in \bar{R}_\tau(T)$ such that

$$\begin{cases}
(D_\tau u_{h,N}(t), v) + d(R_N u_{h,N}(t) + \delta \tau R_N D_\tau u_{h,N}(t), R_N u_{h,N}(t), R_N v) + \mu a(u_{h,N}(t) + \sigma \tau D_\tau u_{h,N}(t), v) - b(v, p_{h,N}(t) + \theta \tau D_\tau p_{h,N}(t)) \\
= (f(t), v), \quad v \in V_{h,N}, t \in \bar{R}_\tau(T), \\
\beta (D_\tau p_{h,N}(t), w) + b(u_{h,N}(t) + \theta \tau D_\tau u_{h,N}(t), w) = 0, \quad \forall w \in W_{h,N}, t \in \bar{R}_\tau(T), \\
u_{h,N}(0) = I_N \cap_{k+1} U_0, \\
p_{h,N}(0) = I_N \cap_{k+1} P_0
\end{cases} \quad (7.6)$$

where $P_0$ is determined by

$$\Delta P_0 = \nabla \cdot (f - (U_0 \cdot \nabla) U_0).$$

If $\delta = \sigma = \theta = 0$, then (7.6) is an explicit scheme. Otherwise a linear iteration is needed to evaluate the values of unknown functions at the interpolation points. In particular, if $\delta = \sigma = \theta = \frac{1}{2}$, then we get from (7.5) that

$$||u_{h,N}(t)||^2 + \|p_{h,N}(t)||^2 + \frac{1}{2} \mu \tau \sum_{s \in \bar{R}_\tau(t-\tau)} |u_{h,N}(s) + u_{h,N}(s + \tau)|^2$$

$$= ||u_{h,N}(0)||^2 + \|p_{h,N}(0)||^2 + \tau \sum_{s \in \bar{R}_\tau(t-\tau)} (I_N \cap_{k+1} f(s), u_{h,N}(s) + u_{h,N}(s + \tau)).$$

Clearly it simulates the conservation (5.17) properly.

We analyze the generalized stability of scheme (7.6). Let $c_I = c_I(k)$ be a positive constant such that for any $v \in V_{h,N},$

$$|v|^2 \leq c_I (h^{-2} + N^2) ||v||^2.$$
Assume that
\[ \sigma > \frac{1}{2}, \quad \text{or} \quad \lambda = \tau \left( h^{-2} + N^2 \right) < \frac{2}{\mu c_I (1 - 2\sigma)}. \]  
(7.7)

Let \( \varepsilon \) be a suitably small positive constant and \( q_0 > 0 \). We consider the following two cases,

\begin{align*}
(i) & \quad 2\theta \sigma \geq \theta + \sigma + 2\varepsilon, \quad 2\theta \geq 1 + 2\varepsilon + q_0, \\
(ii) & \quad 2\theta \sigma \leq \theta + \sigma + 2\varepsilon,
\end{align*}
(7.8)

\[ 2\theta \geq (1 + 2\varepsilon + q_0 + \mu \lambda c_I (\sigma + 2\varepsilon)) \left( 1 - \mu c_I \left( \frac{1}{2} - \sigma \right) \right)^{-1}. \]  
(7.9)

Suppose that \( u_{h,N}(0), p_{h,N}(0), I_N \cap _{k+1} f \) and the right-hand side of the second formula of (7.6) have the errors \( \tilde{u}_0, \tilde{p}, \tilde{f} \) and \( \tilde{g} \), respectively, which induce the errors of \( u_{h,N} \) and \( p_{h,N} \), denoted by \( \tilde{u} \) and \( \tilde{p} \) for simplicity. Then

\[
\begin{cases}
(D_r \tilde{u}(t), v) + d (R_N \tilde{u}(t) + \delta R_N D_r \tilde{u}(t), R_N u_{h,N}(t) + R_N \tilde{u}(t), R_N v) \\
+ d (R_N \tilde{u}(t) + \delta R_N D_r \tilde{u}(t), R_N u_{h,N}(t) + R_N \tilde{u}(t), R_N v) + \mu a (\tilde{u}(t) \\
+ \sigma \tau D_r \tilde{u}(t), v) - b(v, \tilde{p}(t) + \theta D_r \tilde{p}(t)) = \tilde{f}(t), v \in V_{h,N}, t \in \bar{R}_r(T), \\
\beta (D_r \tilde{p}(t), w) + b(\tilde{u}(t) + \theta D_r \tilde{u}(t), w) = \tilde{g}(t), w \in W_{h,N}, t \in \bar{R}_r(T).
\end{cases}
\]

We have the following result.

**Theorem 7.22.** Assume that

(i) (7.7) holds, and (7.8) or (7.9) is valid;

(ii) for certain \( t_1 \in R_r(T) \), \( \rho(t_1) e^{b_1 t_1} \leq \frac{\mu h c(k)}{\tau N (\theta - \delta)^2} \).

Then for all \( t \in R_r(t_1) \),

\[ E(\tilde{u}, \tilde{p}, t) \leq c \rho(t) e^{b_1 t}, \]

\( b_1 \) being some positive constant depending on \( k, \beta, \mu \) and the norm \( \| u_{h,N} \|_{L^2_{[0,T;M^{1.7+1}]}}, \gamma > 0 \).

We know from Theorem 7.22 that if \( \sigma > \frac{1}{2} \), then the restriction on \( \lambda \) disappears and so we can use larger \( \tau \). If \( \theta = \delta > \frac{\sigma}{2\sigma + 1} \) for \( \sigma > \frac{1}{2} \), then the restriction on \( \rho(t) \) also disappears and so the index of generalized stability, \( s = -\infty \). This improves the stability essentially. Indeed, it is the best result. On the other hand, since we adopt the operator \( d(v, w, z) \), the effect of the main nonlinear term is cancelled, i.e.,

\[ d(\tilde{u}(t), \tilde{u}(t), \tilde{u}(t)) = 0. \]

This fact shows again the importance of a reasonable treatment with the leading nonlinear terms in the approximation of nonlinear problems.

We are going to analyze the convergence of scheme (7.6). In order to obtain the uniform optimal error estimation for \( \beta \geq 0 \), we first consider the corresponding Stokes problem, which itself is very important in fluid dynamics. Let \( V' \) be the dual space of \( V = \left( H^1_0, p(\Omega) \right)^2 \) and \( W = L^2_0, p(\Omega) \). \( \langle \cdot, \cdot \rangle \) stands for the duality between \( V' \) and
V. Let \( \eta \in V' \) and \( \xi \in W' = W \). The Stokes problem is to find \( u \in V \) and \( p \in W \) such that

\[
\begin{align*}
\mu a(u, v) - b(v, p) &= \langle \eta, v \rangle, & \forall v \in V, \\
b(u, w) &= \langle \xi, w \rangle, & \forall w \in W;
\end{align*}
\]

(7.10)

Let \( V_{h,N}^* \) be the trial function space of the speed vector \( u \). We approximate \( u \) and \( p \) by \( u^{h,N} \in V_{h,N}^* \) and \( p^{h,N} \in W_{h,N} \) satisfying

\[
\begin{align*}
\mu a(u^{h,N}, v) - b(v, p^{h,N}) &= \langle \eta, v \rangle, & \forall v \in V_{h,N}^*, \\
b(u^{h,N}, w) &= \langle \xi, w \rangle, & \forall w \in W_{h,N}.
\end{align*}
\]

(7.11)

(7.11) is a combined Fourier pseudospectral-finite element approximation to (7.10). Following the same line as in Girault and Raviart (1979), it can be shown that (7.11) has a unique solution provided that the following BB condition (the inf-sup condition) is fulfilled

\[
\sup_{v \in V_{h,N}^*} \frac{b(v, w)}{\|v\|} \geq c\|w\|, \quad \forall w \in W_{h,N}.
\]

(7.12)

This condition is fulfilled for \( V_{h,N}^* = \left( S_{h,k+2,1}^0 \otimes V_N, 2 \right)^2 \), see Canuto, Fujii and Quarteroni (1983). If the components of trial functions belong to the different spaces \( A_{h,N} \) and \( B_{h,N} \), then it is denoted by \( V_{h,N}^* = A_{h,N} \times B_{h,N} \). It was pointed out that (7.12) holds for \( V_{h,N}^* = \left( S_{h,k+2,1}^0 \otimes V_N, 2 \right) \times \left( S_{h,k+1,1}^0 \otimes V_N, 2 \right) \). Guo and Ma (1993) improved this result essentially, stated as in the following lemma.

**Lemma 7.6.** If \( V_{h,N}^* = \left( S_{h,k+2,1}^0 \otimes V_N, 2 \right) \times \left( S_{h,k+1,1}^0 \otimes V_N, 2 \right) \) and (7.10) holds, then (7.12) is fulfilled.

We define the linear operator \( P_* : V \times W \to V_{h,N}^* \times W_{h,N} \) by \( P_*(u, p) = (P_* u, P_* p) = (u^{h,N}, p^{h,N}) \) where \( (u^{h,N}, p^{h,N}) \) is the solution of (7.11). Let \( \tilde{r} = \min(r, k + 2) \). Using Lemma 7.6 and an abstract approximation result in Girault and Raviart (1979), we obtain the following result.

**Lemma 7.7.** Let \( V_{h,N}^* = V_{h,N} \). (7.11) has a unique solution. Moreover if \( u \in M_p^r, s(\Omega) \cap H_{0, p}^1(\Omega), p \in H_{p}^{r-1,s-1}(\Omega) \cap L_{0, p}^2(\Omega) \) and \( r, s \geq 1 \), then

\[
\|u - P_* u\| + \|p - P_* p\| \leq c(k) \left( h^{r-1} + N^{1-s} \right) \left( \|u\|_{M^r, s} + \|p\|_{H^{r-1,s-1}} \right).
\]

If \( u \in H_p^{r, s}(\Omega) \cap H_{0, p}^1(\Omega), p \in H_{p}^{r-1,s-1}(\Omega) \cap L_{0, p}^2(\Omega) \) and \( r, s \geq 1 \), then

\[
\|u - P_* u\| \leq c(k) \left( h^{r} + N^{1-s} \right) \left( \|u\|_{H^r, s} + \|p\|_{H^{r-1,s-1}} \right).
\]

We now deal with the convergence. Let \( U^* = P_* U, P^* = P_* P, \bar{U} = u_{h,N} - U^* \) and
\( \tilde{P} = p_{h,N} - P^* \). We deduce from (6.2) and (7.6) that

\[
\begin{align*}
\begin{aligned}
\left( D_r \tilde{U}(t), v \right) + d \left( R_N \tilde{U}(t) + \delta \tau R_N D_r \tilde{U}(t), R_N U^*(t) + R_N \tilde{U}(t), Rv \right) \\
+ d \left( R_N U^*(t) + \delta \tau R_N D_r U^*(t), R_N \tilde{U}(t), R_N v \right) + \mu a \left( \tilde{U}(t) + \sigma D_r \tilde{U}(t), v \right)
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
- b \left( v, \tilde{P}(t) + \theta D_r \tilde{P}(t) \right) = \sum_{j=1}^{2} G_j (v, t), \\
v \in V_{h,N}, t \in \tilde{R}_r(T),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\beta \left( D_r \tilde{P}(t), w \right) + b \left( \tilde{U}(t) + \theta D_r \tilde{U}(t), w \right) = - (G_8(t), w), \\
v \in W_{h,N}, t \in \tilde{R}_r(T),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\dot{U}(0) = I_N \cap_{k+1} U_0 - U^*(0), \\
\dot{P}(0) = I_N \cap_{k+1} P_0 - P^*(0),
\end{aligned}
\end{align*}
\]

where

\[
G_1(v, t) = (D_r U(t) - \partial_t U(t), v),
\]

\[
G_2(v, t) = (D_r U^*(t) - D_r \tilde{U}(t), v),
\]

\[
G_3(v, t) = d \left( R_N U^*(t), R_N \tilde{U}(t), R_N v \right) - D \left( U(t), U(t), v \right),
\]

\[
G_4(v, t) = \delta \tau d \left( R_N D_r U^*(t), R_N \tilde{U}(t), R_N v \right),
\]

\[
G_5(v, t) = \sigma \mu \tau a \left( D_r U^*(t), v \right),
\]

\[
G_6(v, t) = - \theta b \left( v, D_r P^*(t) \right),
\]

\[
G_7(v, t) = \left( f(t) - I_N \cap_{k+1} f(t), v \right),
\]

\[
G_8(v, t) = \beta D_r P^*(t).
\]

We have the following result.

**Theorem 7.23.** Assume that

(i) (7.4) and condition (i) of Theorem 7.22 are fulfilled;

(ii) for certain constants \( r \geq 1 \) and \( s > \frac{3}{p} \),

\[
U \in C \left( 0, T; M_p^{r,s} \cap H_0^1 \right) \cap H^1 \left( 0, T; H^r \cap H^s \right) \cap H^2 \left( 0, T; H^{-1} \right),
\]

\[
P \in C \left( 0, T; L_0^{r,s} \right) \cap H^1 \left( 0, T; H_{p}^{r-1,s-1} \right),
\]

\[
f \in C \left( 0, T; H_{p}^{r,s} \right);
\]

(iii) for certain \( t_1 \in \tilde{R}_r(T) \) and positive constants \( b_2 \),

\[
e^{b_2 t_1} \left( \tau^2 + h^{2\delta} + N^{-2s} + \beta \right) \leq \frac{\mu \gamma \gamma c(k)}{\tau N(\delta - \theta)t^2}.
\]

Then for all \( t \in \tilde{R}_r(t_1) \),

\[
\left| \left| u_{h,N}(t) - U^*(t) \right| \right|^2 \leq c e^{b_2 t} \left( \tau^2 + h^{2\delta} + N^{-2s} + \beta \right)
\]

where \( b_2 \) is a certain positive constant depending on \( k, \beta, \mu \) and the norms of \( U, P \) and \( f \) in the mentioned spaces.
We find that since we compare \((u_{h,N}, p_{h,N})\) to \((U^*, P^*)\), the solution of the corresponding Stokes equation (7.11), the factor \(\frac{1}{\beta}\) in the error estimates disappears. It improves the error estimates, and covers the special case \(\beta = 0\). If we use this technique for the numerical method discussed in the part 6.3, we can also improve the corresponding results. On the other hand, if we take \(\delta = \theta > \frac{1}{2}\), then \(t_1 = T\). It means the global convergence of scheme (7.6).

The combined Fourier-finite element approximations were first used by Mercier and Raugel (1982). They have been used for various nonlinear problems, such as the steady Navier-Stokes equation in Camuto, Maday and Quarteroni (1984), the unsteady Navier-Stokes equation in Guo and Cao (1992a), Guo and Ma (1991, 1993), and Ma and Guo (1992), the compressible fluid flow in Guo and Cao (1992b). At the same time, the combined Chebyshev-finite element methods developed rapidly, with their applications to nonlinear problems, e.g., see Guo, He and Ma (1995), He and Guo (1995), and Guo, Ma and Hou (1996).
Lecture 8

Spectral Methods on the Spherical Surface

In the previous lectures, we discussed various spectral methods in Descartes coordinates with their applications. However, in meteorology, ocean science, potential magnetostatic field and some other fields, we have to consider problems defined on an elliptic plane, on a spherical surface or in a spherical gap. Several numerical algorithms have been developed for such problems, e.g., see Boyd (1978), Haltiner and Williams (1980), Jarraud and Baede (1985), and Bramble and Pasciak (1985). Some of them are related to the spectral methods on a spherical surface. But there were no rigorous approximation results for a long period. Recently Guo and Zheng (1994), and Guo (1995) developed a framework for the numerical analysis of spectral methods on the spherical surface. We now present some results.

8.1. Spectral Approximation On The Spherical Surface

Let \( \lambda \) and \( \theta \) be the longitude and the latitude respectively. Denote by \( S \) the unit spherical surface,

\[
S = \{ (\lambda, \theta) \mid 0 \leq \lambda < 2\pi, -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} \}.
\]

The differentiations with respect to \( \lambda \) and \( \theta \) are denoted by \( \partial_\lambda \) and \( \partial_\theta \). For a scalar function \( \psi \), some commonly used differential operators are defined as

\[
\nabla \psi = \left( \frac{1}{\cos \theta} \partial_\lambda \psi, \partial_\theta \psi \right),
\]

\[
I(\psi, \varphi) = \frac{1}{\cos \theta} (\partial_\lambda \psi \partial_\theta \varphi - \partial_\theta \psi \partial_\lambda \varphi),
\]

\[
\Delta \psi = \frac{1}{\cos \theta} \partial_\theta (\cos \theta \partial_\theta \psi) + \frac{1}{\cos^2 \theta} \partial^2_\lambda \psi.
\]

Let \( \mathcal{D}(S) \) be the set of all infinitely differentiable functions with the regularity at \( \theta = \pm \frac{\pi}{2} \), and the period \( 2\pi \) for the variable \( \lambda \). \( \mathcal{D}'(S) \) stands for the duality of \( \mathcal{D}(S) \). We define the distributions in \( \mathcal{D}'(S) \) and their derivatives in the usual way. If

\[
\int_S \psi \partial_\lambda \varphi \, dS = -\int_S \varphi \, dS, \quad \forall \, \varphi \in \mathcal{D}(S),
\]

then we say that \( \varphi \) is the derivative of \( \psi \) with respect to \( \lambda \), denoted by \( \partial_\lambda \psi \), etc.. Furthermore, we can define the gradient, the Jacobi operator and the Laplacian in
the sense of distributions. For example, if

\[ \int_S v \Delta w \, dS = \int_S zw \, dS, \quad \forall \ w \in \mathcal{D}(S), \]

then we say that \( z = \Delta v \), etc. Now, let

\[ (v, w) = \int_S vw \, dS, \quad ||v|| = (v, v)^{\frac{1}{2}}, \]

and

\[ L^2(S) = \{ v \in \mathcal{D}'(S) \mid ||v|| < \infty \}. \]

Furthermore, define

\[ H^1(S) = \left\{ v \bigg| \frac{1}{\cos \theta} \partial_\lambda v, \partial_\theta v \in L^2(S) \right\} \]

equipped with the semi-norm and the norm as

\[ ||v||_1 = \left( \left\| \frac{1}{\cos \theta} \partial_\lambda v \right\|^2 + ||\partial_\theta v||^2 \right)^{\frac{1}{2}}, \quad ||v|| = (||v||^2 + ||v||_1^2)^{\frac{1}{2}}. \]

For any positive integer \( r \), we can define the space \( H^r(S) \) with the norm \( || \cdot ||_r \) by induction. In particular, the norm \( || \cdot ||_2 \) is equivalent to \( (||v||^2 + ||\Delta v||^2)^{\frac{1}{2}} \). For any real \( r \geq 0 \), the space \( H^r(S) \) is defined by the interpolation between the spaces \( H^r \) and \( H^{r+1} \). For \( r < 0 \), the space \( H^r(S) \) is the dual of \( H^{-r}(S) \). Particularly, \( H^0(S) = L^2(S) \) and \( ||v||_0 = ||v|| \). It can be verified that for any \( v, w \in H^2(S) \),

\[ -(\Delta v, w) = (\nabla v, \nabla w). \]

We now turn to the \( L^2(S) \)-orthogonal system. Let \( L_l(x) \) be the Legendre polynomial of degree \( l \). The normalized associated Legendre function is given by

\[
\begin{cases}
L_{l,m}(x) = \sqrt{\frac{(2m+1)(m-l)!}{2(m+l)!}} \left( 1 - x^2 \right)^{\frac{1}{2}} \partial_x^m L_m(x), & \text{for } l \geq 0, m \geq |l|, \\
L_{l,m}(x) = L_{-l,m}(x), & \text{for } l < 0, m \geq |l|,
\end{cases}
\]

Moreover, the spherical harmonic function \( Y_{l,m}(\lambda, \theta) \) is

\[ Y_{l,m}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} e^{i\lambda} L_{l,m}(\sin \theta), \quad m \geq |l|. \]

The set of these functions is the \( L^2(S) \)-orthogonal system on \( S \), i.e.,

\[ \int_0^{2\pi} \int_0^\pi Y_{l,m}(\lambda, \theta) \bar{Y}_{l',m'}(\lambda, \theta) \cos \theta d\theta d\lambda = \delta_{ll'}\delta_{mm'}. \]

The spherical harmonic expansion of a function \( v \in L^2(S) \) is

\[ v = \sum_{l=-\infty}^{\infty} \sum_{m \geq |l|} \hat{v}_{l,m} Y_{l,m}(\lambda, \theta) \]
where the coefficient $\hat{v}_{l,m}$ is given by

$$\hat{v}_{l,m} = \frac{2\pi}{2N+1} \left( \sum_{l=-N}^{N} \sum_{m=|l|}^{N} m^r(m+1)^r |\hat{v}_{l,m}|^2 \right)^{\frac{1}{2}}.$$

It can be checked that

$$-\Delta Y_{l,m}(\lambda, \theta) = m(m+1)Y_{l,m}(\lambda, \theta).$$

So $Y_{l,m}(\lambda, \theta)$ is the eigenfunction of the operator $-\Delta$ on $S$, corresponding to the eigenvalue $m(m+1)$. Thus in the space $H^r(S)$, the norm $||v||_r$ is equivalent to

$$\left( \sum_{l=-N}^{N} \sum_{m=|l|}^{N} m^r(m+1)^r |\hat{v}_{l,m}|^2 \right)^{\frac{1}{2}}.$$

Now let $N$ be any positive integer, and define the finite dimensional space $V_N$ as

$$V_N = \text{span} \{ Y_{l,m} \mid |l| \leq N, |m| \leq m \leq M(l) \}.$$

Usually we take $M(l) = N$ or $N + |l|$. For fixing the discussion, we take $M(l) = N$. Let $\tilde{V}_N$ be the subset of $V_N$ containing all real-valued functions, and $c_0$ be a positive constant independent of any function. We have the following results.

**Theorem 8.1.** For any $\phi \in \tilde{V}_N$ and $0 \leq r \leq \mu$,

$$||\phi||_\mu \leq c_0 N^{\mu-r}||\phi||_r.$$

**Theorem 8.2.** For any $v \in H^r(S)$ and $0 \leq \mu \leq r$,

$$||v - P_N v||_\mu \leq c_0 N^{\mu-r}||v||_r.$$

### 8.2. Pseudospectral Approximation On The Spherical Surface

We now present another approximation on the spherical surface, based on the interpolation. Let $x^{(j)}$ and $\omega^{(j)}$ be the Legendre-Gauss interpolation points and the weights on the interval $\Lambda = (-1,1)$. Define $S_N$ as a set of grid points,

$$S_N = \left\{ (\lambda^{(k)}, \theta^{(j)}) \middle| \lambda^{(k)} = \frac{2k\pi}{2N+1}, \theta^{(j)} = \arcsin x^{(j)}, 0 \leq k \leq 2N, 0 \leq j \leq N \right\}.$$

The discrete inner point $(\cdot, \cdot)_N$ and the discrete norm $\| \cdot \|_N$ are defined by

$$(v, w)_N = \frac{2\pi}{2N+1} \sum_{k=0}^{2N} \sum_{j=0}^{N} v(\lambda^{(k)}, \theta^{(j)}) \overline{w}(\lambda^{(k)}, \theta^{(j)}) \omega^{(j)}, \quad ||v||_N = (v, v)^{\frac{1}{2}}.$$

The interpolation $I_N : C(S) \rightarrow V_N$ is such a mapping that for any $v \in C(S)$,

$$I_N v = \sum_{l=-N}^{N} \sum_{m=|l|}^{N} \hat{v}_{l,m} Y_{l,m}(\lambda, \theta).$$
where
\[
\tilde{v}_{i,m} = \frac{2\pi}{2N + 1} \sum_{k=0}^{2N} \sum_{j=0}^{N} v \left( \lambda^{(k)}, \theta^{(j)} \right) \overline{y}_{i,m} \left( \lambda^{(k)}, \theta^{(j)} \right) \omega^{(j)}.
\]

Clearly
\[
\tilde{v}_{i,m} = (v, Y_{i,m})_N.
\]

The degree of freedom of \( S_N \) is \((2N+1)(N+1)\), while the dimension of \( V_N \) is
\[
\sum_{l=0}^{N} (2l+1) = (N+1)^2.
\]
Thus, \( I_N v \left( \lambda^{(k)}, \theta^{(j)} \right) \neq v \left( \lambda^{(k)}, \theta^{(j)} \right) \) generally. Consequently, \( I_N \) is not an interpolation in the usual sense. It is actually the projection from \( C(S) \) onto \( V_N \), corresponding to the discrete inner product \((\cdot, \cdot)_N\). But we have the following results.

**Lemma 8.1.** For any \( v \in C(S) \) and \( \phi \in V_N \), we have that

(i) \( I_N \phi = \phi \),

(ii) \( (I_N v, \phi) = (I_N v, \phi)_N = (v, \phi)_N \).

**Lemma 8.2.** For any \( v \in C(S) \), we have that

(i) \( \|I_N v\| = \|I_N v\|_N \leq \|v\|_N \),

(ii) \( \|v - I_N v\|_N = \inf_{\phi \in V_N} \|v - \phi\|_N \).

**Lemma 8.3.** For any \( \phi \in V_N \), we have that

(i) \( \left\| \frac{1}{\cos \theta} \partial_\lambda \phi \right\|_N = \left\| \frac{1}{\cos \theta} \partial_\lambda \phi \right\| \),

(ii) \( \|\partial_\theta \phi\|_N = \|\partial_\theta \phi\| \).

Further, we get the approximation result as follows.

**Theorem 8.3.** For any \( v \in H^r(S), 0 \leq \mu \leq r \) and \( r > 1 \),

\[
\|v - I_N v\|_\mu \leq c N^{\mu+1+\delta-r} \|v\|_r,
\]

\( \delta \) being an arbitrary positive constant.

All results in this part come from Cao and Guo (1997).

---

**8.3. Applications**

We take the vorticity equation as an example to describe the spectral methods on the surface. Let \( H, \Psi \) and \( \mu > 0 \) be the vorticity, the stream function and the kinetic viscosity, respectively. The vorticity equation on the spherical surface \( S \) is of the form

\[
\begin{aligned}
\partial_t H + J(H, \Psi) - \mu \Delta \Psi &= f, & (\lambda, \theta) &\in S, \ 0 < t \leq T, \\
-\Delta \Psi &= H, & (\lambda, \theta) &\in S, \ 0 \leq t \leq T, \\
H(0) &= H_0, & (\lambda, \theta) &\in S
\end{aligned}
\]
where $f$ and $H_0$ are given functions. It is natural to assume that all functions have the period $2\pi$ for the variable $\lambda$, and are regular at the pole points $\theta = \pm \frac{\pi}{T}$. For fixing $\Psi$, we also require that $\Psi(\lambda, \theta, t) \in L^2_b(S)$ for $0 \leq t \leq T$.

The weak form of (8.1) is to find $H \in L^2(0; T; H^1(S)) \cap L^\infty(0; T; L^2(S))$ and $\Psi \in L^2(0; T; H^1(S) \cap L^2(S))$ such that

\[
\begin{cases}
(\partial_t H(t) + J(H(t), \Psi(t)), v) + \mu(\nabla H(t), \nabla v) = (f(t), v), & \forall v \in H^1(S), 0 < t \leq T, \\
(\nabla \Psi(t), \nabla v) = (H(t), v), & \forall v \in H^1(S), 0 \leq t \leq T, \\
H(0) = H_0,
\end{cases}
\]

If $H_0 \in L^\infty(S)$ and $f \in L^\infty(0, T; L^\infty(S))$, then (8.2) possesses a unique solution and $H \in L^2(0, T; H^1(S)) \cap H^1(0, T, H^{-1}(S))$. The solution is conservative. Indeed, for any $v, z \in H^{r+1}(S), w \in H^1(S)$ and $r > 0$,

\[
(J(v, w), z) + (J(z, w), v) = 0
\]

and so

\[
\|H(t)\|^2 + 2\mu \int_0^t |H(s)|_1^2 \, ds = \|H_0\|^2 + 2 \int_0^t (f(s), H(s)) \, ds.
\]

Now let $V_N^0 = V_N \cap L^2(S)$. $H$ and $\Psi$ are approximated by $\eta_N$ and $\psi_N$. The spectral scheme for (8.2) is to find $\eta_N(\lambda, \theta, t) \in V_N, \psi_N(\lambda, \theta, t) \in V_N^0$ for all $t \in \tilde{R}_\tau(T)$ such that

\[
\begin{cases}
(D_t \eta_N(t) + J(\eta_N(t), \delta \tau D_\tau \eta_N(t), \psi_N(t)), \phi) \\
-\mu(\Delta (\eta_N(t) + \sigma \tau D_\tau \eta_N(t)), \phi) = (f(t), \phi), & \forall \phi \in V_N, t \in \tilde{R}_\tau(T), \\
-(\Delta \psi_N(t), \phi) = (\eta_N(t), \phi), & \forall \phi \in V_N^0, t \in \tilde{R}_\tau(T), \\
\eta_N(0) = P_N H_0, & (\lambda, \theta) \in S
\end{cases}
\]

where $\delta$ and $\sigma$ are the parameters, $0 \leq \delta, \sigma \leq 1$. If both of them are zero, then (8.5) is an explicit scheme. Otherwise a linear iteration is needed at each time $t \in R_\tau(T)$. For any $t \in \tilde{R}_\tau(T), \eta_N(t)$ is determined uniquely. In particular, if $\delta = \sigma = \frac{1}{2}$, then we have from (8.3) and (8.5) that

\[
\|\eta_N(t)\|^2 + \frac{\mu T}{2} \sum_{s \in \tilde{R}_\tau(t-T)} |\eta_N(s) + \eta_N(s + \tau)|_1^2 = \|\eta_N(0)\|^2 + \tau \sum_{s \in \tilde{R}_\tau(t-T)} (f(s), \eta_N(s) + \eta_N(s + \tau)).
\]

This is a reasonable analogy of (8.4).

We now analyze the generalized stability of scheme (8.5). Suppose that $\eta_N(0), f$ and the right-hand side of (8.5) have the errors $\tilde{\eta}_0, \tilde{f}$ and $g$, respectively. They induce the errors of $\eta$ and $\psi$, denoted by $\tilde{\eta}_N$ and $\tilde{\psi}_N$. Then

\[
\begin{cases}
(D_t \tilde{\eta}(t) + J(\tilde{\eta}(t) + \delta \tau D_\tau \eta(t), \psi(t) + \tilde{\psi}(t)) + J(\eta_N(t) + \delta \tau D_\tau \eta(t), \tilde{\psi}(t)), \phi) \\
-\mu(\Delta (\tilde{\eta}(t) + \sigma \tau D_\tau \eta(t)), \phi) = \tilde{f}(t, \phi), & \forall \phi \in V_N, t \in \tilde{R}_\tau(T), \\
-(\Delta \tilde{\psi}(t), \phi) = (\tilde{\eta}(t) + \tilde{g}(t), \phi), & \forall \phi \in V_N^0, t \in \tilde{R}_\tau(T), \\
A \tilde{\psi}(t) = 0, & t \in \tilde{R}_\tau(T), \\
\tilde{\eta}(0) = \tilde{\eta}_0, & (\lambda, \theta) \in S.
\end{cases}
\]
Let \( q_0 \geq 0 \) and \( \varepsilon \) be suitably small positive number. Also set

\[
\begin{align*}
\xi_1 &= \max \left( 1 + q_0 + 3\varepsilon, \frac{2\sigma + 2\varepsilon}{2\sigma - 1} \right), \\
\xi_2 &= 1 + q_0 + 3\varepsilon + \mu c_I \tau N^2 \left( \frac{1}{2} + \varepsilon \right), \\
\xi_3 &= \frac{1 + q_0 + 3\varepsilon + \mu c_I \tau N^2 (\sigma + \varepsilon)}{1 + \mu c_I \tau N^2 (\sigma - \frac{1}{2})},
\end{align*}
\]

where \( c_I \) is a positive constant such that \( \|v\|_1^2 \leq c_I N^2 \|v\|^2 \) for all \( v \in V_N \). Define

\[
\begin{align*}
E(v, t) &= \|v(t)\|^2 + \tau \sum_{s \in \bar{R}_r(t-\tau)} \left( \mu |v(s)|_1^2 + q_0 \tau \|D_r v(s)\|^2 \right), \\
\rho(t) &= \|\hat{\eta}(0)\|^2 + \mu \tau \left( \sigma + \frac{\xi}{2} \right) |\hat{\eta}(0)|_1^2 + \tau \sum_{s \in \bar{R}_r(t-s)} \left( \|\hat{f}(s)\|^2 + \|\hat{g}(s)\|^2 \right).
\end{align*}
\]

We claim the first result as follows.

**Theorem 8.4.** Assume that

(i) \( \tau N^{2\gamma} \) is suitably small, \( \gamma \) being an arbitrary small positive constant;

(ii) \( \sigma > \frac{1}{2} \) or \( \tau N^2 < \frac{2}{c_I \mu (1 - 2\sigma)} \);

(iii) \( \|\hat{g}(t)\|^2 \leq \frac{b_1}{\tau N^{2\gamma}} \) and \( \rho(t_1) e^{b_1 t} \leq \frac{b_3}{\tau N^{2\gamma}} \) for some \( t_1 \in \bar{R}_r(T) \).

Then for all \( t \in \bar{R}_r(t_1) \)

\[
E(\hat{\eta}, t) \leq \rho(t) e^{b_1 t} \tag{8.6}
\]

where \( b_1 - b_3 \) are some positive constants depending on \( \mu \) and \( \|\eta_N\|_{C(0,T;H^{\gamma+1})} \).

We now consider a special case, i.e.,

\[
\begin{align*}
2\delta &\geq \xi_1, & \text{for } \sigma > \frac{1}{2}, \\
2\delta &\geq \xi_2, & \text{for } \sigma = \frac{1}{2}, \\
2\delta &\geq \xi_3, & \text{for } \sigma < \frac{1}{2},
\end{align*}
\]

**Theorem 8.5.** If (8.7) and condition (ii) of Theorem 8.4 are fulfilled, then for all \( \rho(t) \) and \( t \in \bar{R}_r(T) \), the estimate (8.6) is valid.

We know again from Theorem 8.5 that a suitable implicit approximation of non-linear term can improve the stability essentially. Indeed in the case of Theorem 8.5, scheme (8.5) has the generalized stability index \( s = -\infty \).

Finally we deal with the convergence of (8.5). First of all, for \( v \in H^2(S) \),

\[
P_N \Delta v(\lambda, \theta) = \Delta P_N v(\lambda, \theta). \tag{8.8}
\]
Let $H_N = P_N H$, $\Psi_N = P_N \Psi$, $\tilde{\Psi} = \eta_N - H_N$ and $\tilde{\Psi} = \psi_N - \Psi_N$. Then we have from (8.2), (8.5) and (8.8) that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
D_r \bar{H}(t) + J \left( \bar{H}(t) + \delta \tau D_r \bar{H}(t), \Psi_N(t) + \tilde{\Psi}(t) \right) \\
+ J \left( H_N(t) + \delta \tau D_r H_N(t), \tilde{\Psi}(t) \right), \phi - \mu \left( \Delta \bar{H}(t) + \sigma \tau D_r \bar{H}(t) \right), \phi \\
\end{array} \right. \quad - \sum_{j=1}^{3} G_j(t), \phi - (\nabla G_4(t), \nabla \phi), \quad \forall \phi \in V_N, t \in \bar{R}_r(T),

&- \left( \Delta \tilde{\Psi}(t), \phi \right) = \left( \bar{H}(t), \phi \right), \quad \forall \phi \in V_0^\circ, t \in \bar{R}_r(T),

&A \left( \tilde{\Psi}(t) \right) = 0, \quad t \in \bar{R}_r(T),

&\bar{H}(0) = 0, \quad (\lambda, \theta) \in S,
\end{aligned}
\]

where

\[
G_1(t) = D_r H_N(t) - \partial_r H_N(t),
G_2(t) = J \left( H_N(t), \Psi_N(t) \right) - J \left( H(t), \Psi(t) \right),
G_3(t) = \delta \tau J \left( D_r H_N(t), \Psi_N(t) \right),
G_4(t) = \mu \sigma \tau D_r H_N(t).
\]

Guo (1995) proved the following result.

**Theorem 8.6.** Assume that

(i) (8.7) or condition (i) of Theorem 8.4 is valid;
(ii) condition (ii) of Theorem 8.4 is fulfilled;
(iii) For $r > 0$, $H \in C(0,T;H^{r+1}) \cap H^1(0,T;H^1) \cap H^2(0,T;L^2)$.

Then for all $t \in \bar{R}_r(T)$,

\[
E (\eta_N - H_N, t) \leq b_4 \left( \tau^2 + N^{-2r} \right),
\]

$b_4$ being a positive constant depending on $\mu$ and the norms of $H$ in the mentioned spaces.

The spectral methods on the spherical surface have also been used for the barotropic vorticity equation in Guo and Zheng (1994), and for a system of nonlinear partial differential equations governing the fluid flows with low Mach number in Guo and Cao (1995). The applications of the pseudospectral methods on the surface can be found in Cao and Guo (1997).
References


127


