Paley-Wiener Theorem for Dunkl Transform

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For $\alpha \in \mathbb{R}^n \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^n$ orthogonal to $\alpha$. A finite set $R \subset \mathbb{R}^n \setminus \{0\}$ is called a root system if $R \cap R \cdot \alpha = \{\pm \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system $R$ the reflections $\sigma_\alpha$ generate a finite group $W \subset O(n)$. All reflections in $W$ correspond to suitable pairs of roots. For a given $\beta \in R^+ \cup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{ \alpha \in R : (\alpha, \beta) > 0 \}$. We assume the root system $R$ is normalized in the sense that $|\alpha| = \sqrt{2}$ for all $\alpha \in R$. A function $k : R \to C$ on a root system $R$ is called a multiplicity function if it is invariant under the action of the associated reflection group $W$. Denotes the number of conjugacy classes of reflections by $m$. Let $K = C^m$.

The Dunkl operators $T_\zeta, \zeta \in \mathbb{R}^n$, on $\mathbb{R}^n$ associated with the finite reflection group $W$ and multiplicity function $k$ are given by

$$T_\zeta f(x) = \partial_\zeta f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_i \cdot \frac{f(x) - f(\sigma_\alpha)}{\langle \alpha, x \rangle}.$$ 

Consider the system

$$T_\zeta (k) f = (\lambda, \zeta) f. \tag{1}$$

There exists an open set $K^{ref} \subset K$ invariant under complex conjugation and containing $\{ k \in K | \Re(k) \geq 0 \}$ such that the solution space of (1) is 1-dimensional for all $k \in K^{ref}$ and $\lambda \in C^n$. This solution space contains a unique function $Exp_G(\lambda, k, \cdot)$ such that $Exp_G(\lambda, k, 0) = 1$.

Let $\Re(k) \geq 0$. Put

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}.$$ 

The Dunkl transforms are defined as follows

$$(D_k f)(\lambda) = \frac{1}{c_k} \int_{\mathbb{R}^n} f(x) Exp_G(-i\lambda, k, x) w_k(x) dx,$$

$$(E_k f)(x) = \frac{1}{c_k} \int_{\mathbb{R}^n} f(\lambda) Exp_G(i\lambda, k, x) w_k(\lambda) d\lambda.$$ 

The constant $c_k$ is known as a Mehta-type integral. The Plancherel theorem for the Dunkl transforms says that $D_k$ and $E_k$ are unitary operators on $L_2(\mathbb{R}^n, |w_k(x)| dx)$, and they are the inverses of each other.

In this talk we establish a Paley-Wiener-type theorem for the Dunkl transforms of functions with compact support. The characterization is formulated on $\mathbb{R}^n$ without passing to complexification.