A survey of chaotic dynamics (I):
Uniformly Hyperbolic dynamics

Jean-Christophe Yoccoz
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**Definition:** A linear automorphism $T$ of a finite-dimensional vector space is *hyperbolic* if no (complex) eigenvalue of $T$ has modulus 1.

Proposition: A linear automorphism $T$ of a Banach space $E$ is hyperbolic if there exists a continuous splitting $E = E^s \oplus E^u$ into $T$-invariant closed subspaces and constants $C > 0, 0 < \lambda < 1$ such that, for all $n \geq 0$, $||T^n|_{E^s}|| \leq C \lambda^n, ||T^n|_{E^u}|| \leq C \lambda^n$.

Remark: One can find an equivalent norm such that these estimates hold with $C = 1$.
Hyperbolic linear maps

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$$||T^n_{|E_s}|| \leq C\lambda^n, \quad ||T_{|E_u}^{-n}|| \leq C\lambda^n.$$
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Hyperbolic fixed points

Let $E$ be a Banach space, $U$ an open subset, $f : U \to E$ be a $C^1$ map.

Definition: A fixed point $x^*$ of $f$ is hyperbolic if the tangent map $T_{x^*}f$ is a hyperbolic linear automorphism of $E$.

Theorem: (Hartman-Grobman) Let $T$ be a linear hyperbolic automorphism of $E$. There exists $\epsilon > 0$ such that, if $\Delta f : E \to E$ is a Lipschitz bounded map with $\text{Lip}(\Delta f) < \epsilon$, then there exists a unique homeomorphism $h : E \to E$ such that $h - \text{id}$ is bounded and $h \circ T \circ h^{-1} = T + \Delta f$.

Corollary: The dynamics in a neighborhood of a hyperbolic fixed point are topologically conjugated to those of its tangent map.
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**Corollary:** The dynamics in a neighborhood of a hyperbolic fixed point are topologically conjugated to those of its tangent map.
Sternberg’s linearization theorem

**Theorem:** (Sternberg) Assume that $E$ is finite-dimensional, that $x^*$ is a hyperbolic fixed point of $f$, that $T_{x^*}f$ is semi-simple (diagonalizable over $\mathbb{C}$) and there are no *resonances* between the (complex) eigenvalues of $T_{x^*}f$. Then, $f$ is $C^\infty$-linearizable: there is a local $C^\infty$-diffeomorphism $h : (E, x^*) \rightarrow (E, 0)$ such that

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Theorem: (Sternberg) Assume that $E$ is finite-dimensional, that $x^*$ is a hyperbolic fixed point of $f$, that $T_{x^*}f$ is semi-simple (diagonalizable over $\mathbb{C}$) and there are no resonances between the (complex) eigenvalues of $T_{x^*}f$. Then, $f$ is $C^\infty$-linearizable: there is a local $C^\infty$-diffeomorphism $h : (E, x^*) \to (E, 0)$ such that

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A resonance is a relation

$$\lambda_i = \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \quad 1 \leq i \leq n, \quad j_m \geq 0, \quad \sum_{1}^{n} j_m \geq 2,$$

between the eigenvalues of $T_{x^*}f$. 

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between the eigenvalues of $T_{x^*}f$.

Remark: There is no "diophantine" hypothesis, i.e no "small divisor" problem.
The stable manifold theorem

Let \( E \) be a Banach space, \( T \) be a linear automorphism of \( E \). Let \( 0 < \kappa_s < \kappa_u \) be such that the spectrum of \( T \) is disjoint from the annulus \( \{ \kappa_s \leq |\lambda| \leq \kappa_u \} \), and let \( E = E_s \oplus E_u \) be the associated decomposition into \( T \)-invariant subspaces.
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**Theorem:** Let $\kappa \in (\kappa_s, \kappa_u)$. Let $f : E \to E$ be a Lipschitz map satisfying $f(0) = 0$ and $\text{Lip}(f - T) < \min(\kappa - \kappa_s, \kappa_u - \kappa)$. Then the set

$$W^s_{\kappa}(f) := \{x \in E, \sup_{n \to +\infty} \kappa^{-n} ||f^n(x)|| < +\infty\}$$

is the graph of a contracting map $g : E_s \to E_u$ satisfying $g(0) = 0$. For $x \in W^s_{\kappa}(f)$, one has

$$\lim_{n \to +\infty} \kappa^{-n} ||f^n(x)|| = 0.$$
Smoothness of the stable manifold

**Remark:** The set $W^s_\kappa(f)$ does not depend on $\kappa$, provided $\text{Lip}(f - T) < \min(\kappa - \kappa_s, \kappa_u - \kappa)$. 

**Terminology:**
- The set $W^s_\kappa(f)$ is called a **strong stable manifold** when $\kappa_u \leq 1$,
- a **center-stable manifold** when $\kappa_s \geq 1$,
- and the **stable manifold** when $\kappa_s < 1 < \kappa_u$.

**Proposition:** When $\kappa < 1$ and $f$ is $C^r$ ($r$ real $\geq 1$, $r = \omega$), then $g$ is $C^r$. Moreover, $D_0 g = 0$ if $D_0 (f - T) = 0$.

**Remark:** When $\text{Lip}(f - T)$ is small enough, $f$ is a biLipschitz homeomorphism and one can apply the theorem to $f^{-1}$. 

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**Remark:** The set $W_{\kappa}^{s}(f)$ does not depend on $\kappa$, provided $\text{Lip}(f - T) < \min(\kappa - \kappa_s, \kappa_u - \kappa)$.

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Let $K$ be a compact metric space, $f$ be a homeomorphism of $K$, $p : E \to K$ be a Banach vector bundle over $K$, and $F : E \to E$ be an automorphism of $E$ over $f$. 

Proposition: The following are equivalent

1. The automorphism $\sigma \mapsto F \circ \sigma \circ f^{-1}$ induced by $F$ on the space of continuous sections of $p$ is hyperbolic.

2. The automorphism $\sigma \mapsto F \circ \sigma \circ f^{-1}$ induced by $F$ on the space of bounded sections of $p$ is hyperbolic.

3. There exists a continuous splitting $E = E_s \oplus E_u$ into closed $F$-invariant subbundles and constants $C > 0$, $0 < \lambda < 1$ such that, for all $n \geq 0$,

$$||F^n||_{E_s} \leq C \lambda^n,$$

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When these conditions are satisfied, we say that $F$ is hyperbolic.

The splitting of the spaces of bounded/continuous sections are induced by the splitting of $E$. 

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Definition: Let $M$ be a manifold, $K$ be a compact subset, $U$ a neighborhood of $K$, $f : U \rightarrow M$ a $C^1$-embedding such that $f_{|K}$ is a homeomorphism of $K$.
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Remark: The splitting $E = E_1 \oplus E_2$ is not assumed to be invariant.
Let $K$ be a compact metric space, $f$ be a homeomorphism of $K$, $p : E \to K$ be a Banach vector bundle over $K$, and $F : E \to E$ be an automorphism of $E$ over $f$. Let $E_x = E_{1,x} \oplus E_{2,x}$ be a (not necessarily continuous) splitting of each fiber into subspaces equipped with norms $\| \cdot \|_{i,x}$, $i = 1, 2$, satisfying, for some $C > 1$ and any $x \in K$, any $v_1 \in E_{1,x}$, $v_2 \in E_{2,x}$

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C^{-1} \max(\|v_1\|_{1,x}, \|v_2\|_{2,x}) \leq \|v_1 + v_2\| \leq C \max(\|v_1\|_{1,x}, \|v_2\|_{2,x}).
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The conefield criterion (I)

Let $K$ be a compact metric space, $f$ be a homeomorphism of $K$, $p : E \to K$ be a Banach vector bundle over $K$, and $F : E \to E$ be an automorphism of $E$ over $f$.

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For $\lambda \in \mathbb{R}_+$, let

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be the cones at $x$ of slope $\lambda$ associated to this decomposition.
The cone field criterion (I)

Let $K$ be a compact metric space, $f$ be a homeomorphism of $K$, $p : E \to K$ be a Banach vector bundle over $K$, and $F : E \to E$ be an automorphism of $E$ over $f$.

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be the cones at $x$ of slope $\lambda$ associated to this decomposition.

**Remark:** The splitting $E = E_1 \oplus E_2$ is **not** assumed to be invariant.
Proposition: In this setting, assume there exists $0 < \lambda < 1$, $\mu > 1$ and an integer $m \geq 1$ such that

1. for any $x \in K$, one has $F(C \lambda - 1(x)) \subset C(\lambda f(x))$ (this implies $F^{-1}(C^* \lambda - 1(x)) \subset C^*(\lambda f(x)))$);
2. for any $x \in K$, one has $E_x = F(E_{1,x} f^{-1}(x)) \oplus E_{2,x}$, $x = E_{1,x} \oplus F^{-1}(E_{2,x} f(x))$);
3. for any $x \in K$, $v \in C(\lambda - 1(x))$, $||F_m(v)|| \geq \mu ||v||$;
4. for any $x \in K$, $v \in C^*(\lambda - 1(x))$, $||F^{-m}(v)|| \geq \mu ||v||$.

Then, $F$ is hyperbolic and $E_u(x) \subset C(\lambda x)$, $E_s(x) \subset C^*(\lambda x)$ for all $x \in K$. 
The conefield criterion (II)

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Then, $F$ is hyperbolic and $E_* u(x) \subset C_{\lambda}(x)$, $E_s(x) \subset C^*_{\lambda}(x)$ for all $x \in K$. 

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3. for any $x \in K$, $v \in C_{\lambda-1}(x)$, $\|F^m(v)\|_{f^m(x)} \geq \mu \|v\|_x$;

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Then, $F$ is hyperbolic and $E_u(x) \subset C_{\lambda}(x)$, $E_s(x) \subset C^*_{\lambda}(x)$ for all $x \in K$. 

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Proposition: In this setting, assume there exists $0 < \lambda < 1$, $\mu > 1$ and an integer $m \geq 1$ such that

1. for any $x \in K$, one has $F(C_{\lambda^{-1}}(x)) \subset C_{\lambda}(f(x))$ (this implies $F^{-1}(C^*_{\lambda^{-1}}(x)) \subset C^*_\lambda(f(x)))$;

2. for any $x \in K$, one has $E_x = F(E_{1,f^{-1}(x)}) \oplus E_{2,x} = E_{1,x} \oplus F^{-1}(E_{2,f(x)});

3. for any $x \in K$, $v \in C_{\lambda^{-1}}(x)$, $\|F^m(v)\|_{f^m(x)} \geq \mu \|v\|_x$;

4. for any $x \in K$, $v \in C^*_{\lambda^{-1}}(x)$, $\|F^{-m}(v)\|_{f^{-m}(x)} \geq \mu \|v\|_x$.

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A survey of chaotic dynamics (I): Uniformly Hyperbolic dynamics
Let $M$ be a manifold, $K$ be a compact subset, $U$ a neighborhood of $K$, $f : U \rightarrow M$ a $C^1$-embedding such that $f|_K$ is a homeomorphism of $K$. 

Corollary: Assume that $K$ is hyperbolic. Then there exists a compact neighborhood $W \subset U$ of $K$ such that the maximal compact invariant set $\bigcap_{n \in \mathbb{Z}} f^{-n}(W) \supset K$ is also hyperbolic.

Definition: An invariant compact set $K \subset U$ is locally maximal if there exists a compact neighborhood $W \subset U$ of $K$ such that $\bigcap_{n \in \mathbb{Z}} f^{-n}(W) = K$. 

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**Definition:** An invariant compact set $K \subset U$ is *locally maximal* if there exists a compact neighborhood $W \subset U$ of $K$ such that $\bigcap_{n \in \mathbb{Z}} f^{-n}(W) = K$. 
**Definition:** A $C^1$-diffeomorphism of a compact connected manifold $M$ is an *Anosov diffeomorphism* if the (compact invariant) set $M$ is hyperbolic.
Definition: A $C^1$-diffeomorphism of a compact connected manifold $M$ is an Anosov diffeomorphism if the (compact invariant) set $M$ is hyperbolic. Anosov diffeomorphisms form an open (frequently empty!) subset of $\text{Diff}^1(M)$. 

Let $A \in \text{SL}(d, \mathbb{Z})$ be a hyperbolic matrix. It preserves $\mathbb{Z}^d$ hence induces a diffeomorphism of $\mathbb{T}^d$, which is Anosov. 

Theorem: (Franks) Let $A \in \text{SL}(d, \mathbb{Z})$ be hyperbolic and $f \in \text{Diff}^1(\mathbb{T}^d)$ be homotopic to $A$. Then $f$ is semi-conjugate to $A$: there exists a unique continuous surjective map $h: \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity such that $h \circ f = A \circ h$. If $f$ is Anosov, $h$ is a homeomorphism. Conversely, any Anosov diffeomorphism of the torus is homotopic to a linear one.
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Examples (II): The solenoid

Let $D = \{ z \in \mathbb{C}, |z| < 1 \}$ and $f : T \times D \to T \times D$ be defined by

$$f(\theta, z) = (2\theta, \frac{1}{2} \exp(2\pi i \theta) + \frac{1}{4} z).$$
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Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ and $f : \mathbb{T} \times \mathbb{D} \to \mathbb{T} \times \mathbb{D}$ be defined by

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The maximal $f$-invariant set in $\mathbb{T} \times \mathbb{D}$ is the solenoid

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The projection $(\theta, z) \mapsto \theta$ from $S$ to $\mathbb{T}$ is a semi-conjugacy between $f|_S$ and the doubling map $\theta \mapsto 2\theta$ on $\mathbb{T}$. 
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Examples (III): Smale’s horseshoe (in the Hénon family)

The Hénon family (Hénon, 1968) is the 2-parameter family of diffeomorphisms of $\mathbb{R}^2$ defined by

$$H_{b,c}(x, y) := (x^2 + c - by, x).$$
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$$H_{b,c}^{-1}(x, y) := (y, b^{-1}(y^2 + c - x)).$$

Observe that $H_{b,c}$ has constant Jacobian $b$. 

Denote by $K_{b,c}$ the set of points with a bounded orbit. It is a compact (exercise) invariant set. Fix $b > 0$ and consider large negative values for the second parameter $c$. For such values $K_{b,c} = \bigcap_{n \in \mathbb{Z}} H_{b,c}^{-n}(\mathbb{R}^2)$, where $\mathbb{R}^2$ is the square $[-(2|c|)^{1/2}, (2|c|)^{1/2}]^2$. 

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One has

\[ H_{b,c}^{-1}(R) = \{ |x| \leq (2|c|)^{\frac{1}{2}}, |x^2 + c - by| \leq (2|c|)^{\frac{1}{2}} \}, \]

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hence, for \(|c|\) large enough, \( R \cap H_{b,c}^{-1}(R) \) has two ”vertical-like” components in which \(|x| \geq (\frac{1}{2}|c|)^{1/2} \).
One has

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H^{-1}_{b,c}(R) = \{ |x| \leq (2|c|)^{\frac{1}{2}}, |x^2 + c - by| \leq (2|c|)^{\frac{1}{2}} \},
\]

\[
H_{b,c}(R) = \{ |y| \leq (2|c|)^{\frac{1}{2}}, |y^2 + c - x| \leq (2|c|)^{\frac{1}{2}} b \},
\]

hence, for \(|c|\) large enough, \(R \cap H^{-1}_{b,c}(R)\) has two "vertical-like" components in which \(|x| \geq (\frac{1}{2}|c|)^{\frac{1}{2}}\), which are sent by \(H_{b,c}\) onto the two "horizontal-like" components of \(R \cap H_{b,c}(R)\) in which \(|y| \geq (\frac{1}{2}|c|)^{\frac{1}{2}}\).
One has

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As the jacobian matrices of \( H_{b,c}, H_{b,c}^{-1} \) are respectively

\[
\begin{pmatrix} 2x & -b \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -b^{-1} & 2b^{-1}y \end{pmatrix}
\]

the conefield criterion shows that \( K_{b,c} \) is hyperbolic.
**Definition:** A homeomorphism $f$ of a compact metric space $(X, d)$ is *expansive* if there exists $\varepsilon > 0$ such that, for all distinct $x, y \in X$, one has $d(f^n(x), f^n(y)) > \varepsilon$ for some $n \in \mathbb{Z}$.

**Proposition:** Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. Then the restriction of $f$ to $K$ is expansive.
Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. Theorem: There exist a neighborhood $U$ of $f$ in $C^1(U, M)$ and a continuous map $H$ from $U$ to $C(K, M)$ such that ▶ for any $g \in U$, the set $K_g := H(g)(K)$ is a hyperbolic compact invariant set for $g$, satisfying $K_g \circ H(g) = H(g) \circ f$; ▶ for any $g \in U$, $H(g)$ is injective and is the unique map close to the identity satisfying the relation above (in particular $H(f)$ is the identity).
Hyperbolic continuation

Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$.

**Theorem:** There exist a neighborhood $U$ of $f$ in $C^1(U, M)$ and a continuous map $H$ from $U$ to $C(K, M)$ such that

1. For any $g \in U$, the set $K_g := H(g)(K)$ is a hyperbolic compact invariant set for $g$, satisfying $K_g \circ H(g) = H(g) \circ f$;
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Let $(X, d)$ be a compact metric space and let $f$ be a homeomorphism of $X$.
Pseudo-orbits and the chain recurrent set

Let \((X, d)\) be a compact metric space and let \(f\) be a homeomorphism of \(X\).

**Definitions:** Let \(\delta > 0\). A \(\delta\)-pseudo-orbit for \(f\) is a sequence \((x_n)_{n \in \mathbb{Z}}\) in \(X\) such that

\[
d(f(x_n), x_{n+1}) < \delta, \quad \forall n \in \mathbb{Z}.
\]

A point \(x \in X\) is chain-recurrent if, for any \(\delta > 0\), there exists a periodic \(\delta\)-pseudo-orbit through \(x\).

The chain-recurrent set \(R(f)\) is the set of chain-recurrent points. It is compact and \(f\)-invariant.

Remark: Chain-recurrence is the weakest of several notions of recurrence in dynamics.
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The shadowing lemma

**Theorem:** Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. There exists $C > 0$ such that, for any $\delta$-pseudo-orbit $(x_n)_{n \in \mathbb{Z}}$ in $K$, there exists an orbit $(f^n(x))_{n \in \mathbb{Z}}$ shadowing it within $C\delta$:

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1. If $\delta$ is small enough, the shadowing orbit is unique by expansivity.
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**Remarks:**

1. If $\delta$ is small enough, the shadowing orbit is unique by expansivity.

2. In general, the shadowing orbit is **not** contained in $K$, only in a neighborhood of $K$ (and therefore in a larger hyperbolic compact invariant set containing $K$).
Local stable manifolds

Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. 

There exist Riemannian metrics on $U$, with associated distance $d$, such that the local stable manifolds $W_{s_{loc}}(x)$ defined for $x \in K$ by $W_{s_{loc}}(x) = \{ y \in U, \ d(f^n(y), f^n(x)) < 1, \forall n \geq 0 \}$ have the following properties:

▶ for $y \in W_{s_{loc}}(x)$, $n \geq 0$, one has $d(f^n(y), f^n(x)) \leq \kappa^n d(y, x)$ for all $n \geq 0$; in particular, one has $f(W_{s_{loc}}(x)) \subset W_{s_{loc}}(f(x));$

▶ for $x \in K$, $W_{s_{loc}}(x)$ is the image of a $C^1$-embedding $j_x : \{ v \in B_1(E_s, x) \} \to U$ depending continuously on $x$ and satisfying $j_x(0) = x, \quad T_0 j_x = \text{id}_{E_s}.$

Moreover, when $f$ is of class $C^r$ ($r$ integer $\geq 1,$ $r = \infty,$ $r = \omega$), then $j_x$ is also $C^r$ and depends continuously on $x$ in the $C^r$-topology.
Local stable manifolds

Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. Let $0 < \kappa' < \kappa < 1$ be such that, for some $C > 0$, one has $\|Tf^n|_{E_s}\| \leq C\kappa'^n$ for all $n \geq 0$. 
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Jean-Christophe Yoccoz

A survey of chaotic dynamics (I): Uniformly Hyperbolic dynamics
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Moreover, when $f$ is of class $C^r$ ($r$ integer $\geq 1$, $r = \infty$, $r = \omega$), then $j_x$ is also $C^r$ and depends continuously on $x$ in the $C^r$-topology.
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Clearly, one has

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_{loc}(f^n(x)))$$
and it follows that

- for any $y \in W^s(x)$,

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- the stable manifold \( W^s(x) \) is the image of an injective immersion of \( E_{s,x} \), of class \( C^r \) if \( f \) is of class \( C^r \), depending continuously on \( x \) on compact subsets.
Local product structure

Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f : U \to M$. 

Proposition: The following are equivalent:

1. $K$ is locally maximal;
2. $K$ has local product structure: for $x, y \in K$ close enough, the (unique, transverse) point of intersection of $W_{s\text{loc}}(x)$ and $W_{u\text{loc}}(y)$ belongs to $K$;
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Spectral decomposition

**Definition:** A homeomorphism $f$ of a compact metric space $X$ is *topologically mixing* (resp. *transitive*) if, for any nonempty open subsets $U, V$ of $X$, one has $f^n(U) \cap V \neq \emptyset$ for all (resp. some) $n$ large enough.

**Theorem:** (Smale) Let $K$ be a hyperbolic compact invariant set for a $C^1$-embedding $f: U \to M$. Assume that $K$ is locally maximal and $f|_K$ is chain-recurrent. Then there exists a partition $K = \bigoplus_{i=1}^\ell \bigoplus_{j \in \mathbb{Z}} s_i K_{i,j}$ into compact subsets such that $f(K_{i,j}) = K_{i,j} + 1$, $\forall 1 \leq i \leq \ell, j \in \mathbb{Z}$, and the restriction of $f$ to $K_{i,j}$ is topologically mixing. (This implies that the restriction of $f$ to the hyperbolic compact invariant set $\bigoplus_{j \in \mathbb{Z}} s_i K_{i,j}$ is transitive.)
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Periodic points are dense in any basic set.

Definition: A basic set $K$ is an attractor if $K = \bigcap_{n \geq 0} f^n(W)$ for some neighborhood $W$ of $K$ (equivalently, $W_{u_{loc}}(x) \subset K$ for all $x \in K$), a repellor if it is an attractor for $f^{-1}$, and is of saddle-type if it is neither an attractor nor a repellor. Examples of basic sets are $T_d$ (for an Anosov diffeomorphism of the torus), the solenoid (an attractor) or the horseshoe (of saddle-type).
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**Question:** Is an Anosov diffeomorphism always chain-recurrent?

Let $f$ be hyperbolic, and 

$$R(f) = \bigsqcup_{\alpha} R_{\alpha}$$

be the decomposition in basic sets given by the spectral decomposition theorem.
There are associated partitions

\[ M = \bigsqcup_{\alpha} W^s(R_\alpha) = \bigsqcup_{\alpha} W^u(R_\alpha) \]

where

\[ W^s(R_\alpha) \colonequals \{ y \in M, \lim_{n \to +\infty} d(f^n(y), R_\alpha) = \bigsqcup_{x \in R_\alpha} W^s(x) \} \]

and similarly for \( W^u(R_\alpha) \).

The relation \( R_\alpha \sqsubseteq R_\beta \iff W^u(R_\alpha) \cap W^s(R_\beta) \neq \emptyset \) has no cycle of length > 1, hence can be minimally completed as a partial order on the set of basic sets. The attractors (resp. repellors) are the minimal (resp. maximal) elements of this partial order.
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Symbolic dynamics: the shift map

Let $\mathcal{A}$ be a finite alphabet. The homeomorphism $\sigma$ of $\mathcal{A}^\mathbb{Z}$ defined for $\underline{\theta} = (\theta_n)_{n \in \mathbb{Z}}$ by
\[ (\sigma(\underline{\theta}))_n = \theta_{n+1} \]
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The same formula defines a continuous surjective map $\sigma_+$ from $A^{\mathbb{Z}^+}$ to itself, called the unilateral full shift on $A$. 

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The restriction of $\sigma$ (or $\sigma_+$) to a closed invariant subset is called a subshift.
Let $\mathcal{B}$ be a subset of $A \times A$. The subset

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\Sigma_{\mathcal{B}} := \{ \theta \in A^\mathbb{Z}, (\theta_n, \theta_{n+1}) \in \mathcal{B}, \forall n \in \mathbb{Z} \}
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is closed and invariant under $\sigma$; the restriction of $\sigma$ to $\Sigma_{\mathcal{B}}$ is called the \textit{subshift of finite type} defined by $\mathcal{B}$.
Subshifts of finite type

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One defines similarly $\Sigma_{\mathcal{B}}^+ \subset A^{\mathbb{Z}+}$. 
Let $B$ be a subset of $A \times A$. The subset

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**Example:** Let $d \geq 1$ and $A' := A^d$. The map

$$i = i_d : A^\mathbb{Z} \rightarrow (A')^\mathbb{Z}$$

defined by

$$(i(\theta))_n = (\theta_n, \ldots, \theta_{n+d-1}) \in A'$$

is a conjugacy between the full shift on $A$ and a subshift of finite type on $A'$. 
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A characterization of subshifts of finite type

Proposition: Let $\Sigma$ be a closed $\sigma$-invariant subset of $A^\mathbb{Z}$. The following are equivalent:

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2. there exists $d \geq 1$ such that $i_d(\Sigma)$ is a subshift of finite type of $(A^d)^\mathbb{Z}$.
Transitivity and topological mixing for subshifts of finite type

The full shift $\sigma$ is a topologically mixing (hence transitive) homeomorphism of $A^\mathbb{Z}$. 

Let $B \subset A \times A$. When is the subshift of finite type associated to $B$ transitive, topologically mixing? To formulate the answer, we introduce the graph $\Gamma_B$ whose set of vertices is $A$ and which has one arrow from $a$ to $a'$ for each $(a, a') \in B$. We also introduce the matrix $A_B$ indexed by $A \times A$ defined by:

$$A_B^a, a' = \begin{cases} 1 & \text{if } (a, a') \in B \\ 0 & \text{otherwise} \end{cases}$$

For $m \geq 0$, the entry in position $(a, a')$ in $A_B^m$ is the number of oriented paths of length $m$ from $a$ to $a'$ in $\Gamma_B$. 

Jean-Christophe Yoccoz

A survey of chaotic dynamics (I): Uniformly Hyperbolic dynamics
Transitivity and topological mixing for subshifts of finite type

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one has $\sigma(A_i) = A_{i+1}$ and $\sigma_s|_{\Sigma_i}$ is topologically mixing. The period is the g.c.d. of the lengths of the loops in $\Gamma_B$. 
Proposition: Let $K$ be a basic set for a $C^1$-embedding $f : U \to M$.

Idea of proof: As $K$ is totally discontinuous, one can find a finite partition $K = \bigsqcup_{\alpha \in A} K_\alpha$ into clopen subsets with arbitrary small diameter. For $x \in K$, define $h(x) = \theta \in A$ by $	heta_n = \alpha \iff f^n(x) \in K_\alpha$. The map $h$ is continuous, injective (by expansivity), and satisfies $h \circ f = \sigma \circ h$. It is therefore a conjugacy from $f|_K$ to some subshift. As $K$ is locally maximal, the same is true for $h(K)$, which is therefore a subshift of finite type.

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**Totaly discontinuous basic sets**

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