

A Constrained Minimization Problem and its Application to Guided Cylindrical TM-modes in a Homogeneous Self-Focusing Dielectric

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Abstract

This paper is divided into two parts. The first deals with a problem of constrained minimization of an integral functional. In the second part, this is used to establish the existence of guided modes in an isotropic nonlinear dielectric medium.

Part I deals with the following problem:

$$\min\{F(\eta) : \eta \in H_0^1(0, \infty) \text{ and } \int_0^\infty \eta^2 dr = d^2\} \quad (0.1)$$

where

$$F(\eta) = \int_0^\infty r \Psi \left(\frac{\eta}{\sqrt{r}}, \frac{1}{\sqrt{r}} [\eta' + \frac{\eta}{2r}] \right) dr \text{ and } d > 0. \quad (0.2)$$

The essential features of the integrand are that

$$\Psi : \mathbf{R} \rightarrow \mathbf{R} \text{ is convex, whereas} \quad (0.3)$$

$$\psi : [0, \infty)^2 \rightarrow \mathbf{R} \text{ is concave, where } \psi(\mathbf{s}_1, \mathbf{s}_2) = \Psi(\sqrt{2\mathbf{s}_1}, \sqrt{2\mathbf{s}_2}). \quad (0.4)$$

Our main results are Theorem 4.3 which gives conditions ensuring that the minimum is attained by a function satisfying the constraint and Corollary 5.2 which

shows that the Euler-Lagrange equation is satisfied in the classical sense by the minimizer which is of class C^2 . Useful estimates for the Lagrange multiplier and information about the bifurcation of the minimizers are also obtained.

One motivation for the study of integrands with this structure comes from a fundamental problem concerning guided waves in nonlinear optics. Mathematically, this amounts to finding solutions of Maxwell's equations with a nonlinear constitutive relation in the form of traveling waves with finite energy. Solutions in one such family are called TM-modes since the magnetic field is everywhere perpendicular to the direction of propagation. In Part II, we show how this problem can be reduced to a variational form of the kind treated in Part I. The main step involves expressing the constitutive relation for the medium in a dual form and then showing that this translates the natural properties of a self-focusing dielectric material to the essential features (0.3) and (0.4) of the integrand required in Part I. For our discussion of this problem we take Maxwell's equations in the form:

$$\begin{cases} \partial_t \mathbf{B} = -c \nabla \wedge \mathbf{E}, & \partial_t \mathbf{D} = c \nabla \wedge \mathbf{H} \\ \nabla \cdot \mathbf{B} = 0, & \nabla \cdot \mathbf{D} = 0 \end{cases} \quad (0.5)$$

where $c > 0$ is the speed of light in a vacuum, with the constitutive relations that $\mathbf{H} \equiv \mathbf{B}$ and

$$\begin{bmatrix} \varepsilon_1(\langle E_T \rangle^2, \langle E_z \rangle^2) & 0 & 0 \\ 0 & \varepsilon_1(\langle E_T \rangle^2, \langle E_z \rangle^2) & 0 \\ 0 & 0 & \varepsilon_2(\langle E_T \rangle^2, \langle E_z \rangle^2) \end{bmatrix} \quad (0.6)$$

where $\langle E_T \rangle^2$ and $\langle E_z \rangle^2$ denote the time-averages of $|\mathbf{E} \wedge i_z|^2$ and $|\mathbf{E} \cdot i_z|^2$. This form of dielectric response is appropriate for anisotropic, but uni-axial materials, having the z -axis as axis of symmetry where i_z is a unit vector in the direction of that axis. This symmetry means that it is possible to seek solutions with the direction of the z -axis. For TM-modes the magnetic field should then be in the form

$$\mathbf{B} = w(r) \cos(kz - \omega t) i_\theta \quad (0.7)$$

where (r, θ, z) are the usual cylindrical polar co-ordinates and $i_\theta = (-\sin\theta, \cos\theta, 0)$.

In Part II, we show how the problem of finding solutions in this form can be reduced to (0.1). The constitutive relation (0.4) involves the function

$$\Phi(s_1, s_2) = \begin{pmatrix} \varepsilon_1(\frac{1}{2}s_1^2, \frac{1}{2}s_2^2)s_1 \\ \varepsilon_2(\frac{1}{2}s_1^2, \frac{1}{2}s_2^2)s_2 \end{pmatrix} \quad (0.8)$$

and we are led to the functional (0.2) where Ψ is the Legendre transform of Φ . The properties (0.3) and (0.4) are deduced from the properties of ε_1 and ε_2 which characterize a self-focusing dielectric.