Smoothing property for Schrödinger equations with potential superquadratic at infinity

Kenji Yajima and Guoping Zhang
Department of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

I. Assumptions

The present paper is concerned with the smoothing property of the propagator e^{-itH} associated with the initial value problem for the one dimensional Schrödinger equation in the Hilbert space $L^2(\mathbb{R})$:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = Hu \equiv (-\Delta + V(x))u, & x \in \mathbb{R}^1,\ t \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^1, \end{cases} \tag{0.1}$$

Assumption 0.1 *$V(x)$ is a real valued C^3-function of $x \in \mathbb{R}$. There exits a constant $R > 0$ such that the following conditions are satisfied for $|x| \geq R$ with $k > 2$:*

(V1) *$V(x)$ is convex.*

(V2) *For $j = 1, 2, 3$, $|V^{(j)}(x)| \leq C_j \langle x\rangle^{-1} |V^{(j-1)}(x)|$ for some constants C_j.*

(V3) *For positive constants D_1 and D_2, $D_1\langle x\rangle^k \leq V(x) \leq D_2\langle x\rangle^k$.*

II. Smoothing Property of Linear Operator

Under Assumption 0.1 the operator $H = -\Delta + V(x)$ defined on $C_0^\infty(\mathbb{R})$ is essentially selfadjoint in $L^2(\mathbb{R})$ and we denote its closure by the same symbol. Then, H is selfadjoint with the maximal domain $D(H) = \{u \in L^2(\mathbb{R}) : -\Delta u(x) + V(x)u(x) \in L^2(\mathbb{R})\}$ and the solution of (0.1) is given in terms of the exponential function of H by $u(t,x) = e^{-itH}u_0(x)$. H is bounded from below and its spectrum consists only of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots \to \infty$. We denote the corresponding normalized eigenfunctions by $\psi_n(x)$, $n = 1, 2, \ldots$. We set for $2 \le p \le \infty$ and $2 < k < \infty$:

$$\theta(k,p) = \begin{cases} \dfrac{1}{k}\left(\dfrac{1}{2} - \dfrac{1}{p}\right), & \text{if} \quad 2 \le p < 4; \\ \left(\dfrac{1}{4k}\right)_-, & \text{if} \quad p = 4; \\ \dfrac{1}{4} - \dfrac{1}{3}\left(1 - \dfrac{1}{p}\right)\left(1 - \dfrac{1}{k}\right), & \text{if} \quad 4 < p \le \infty, \end{cases}$$

where a_- denotes any number $< a$.

Theorem 0.2 *Suppose V satisfies* Assumption 0.1*. Let $2 \le p \le \infty$ and let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta \le \theta(k,p)$. Then there exists a constant $C > 0$ such that*

$$\|g(t)\langle i\partial/\partial t\rangle^\alpha \langle H\rangle^\beta e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^2(\mathbb{R}_t))} \le C\|g\|_{H^{\frac{1}{2}}(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R}_x)}, \qquad (0.2)$$

for any $g \in H^{\frac{1}{2}}(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$.

Theorem 0.3 *Suppose V satisfies* Assumption 0.1*. Let $K \subset \mathbb{R}$ be compact and let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta \le \dfrac{1}{2k}$. Then, there exists a constant $C > 0$ such that*

$$\sup_{x \in K} \int_{-\infty}^{\infty} |g(t)\langle i\partial/\partial t\rangle^\alpha \langle H\rangle^\beta e^{-itH}u_0(x)|^2 dt \le C\|g\|^2_{H^{\frac{1}{2}}(\mathbb{R}_t)}\|u_0\|^2_{L^2(\mathbb{R}_x)} \qquad (0.3)$$

for any $g \in H^{\frac{1}{2}}(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$.

III. Application to NLS

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -\triangle u + V(x)u + f(x,u), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{0.4}$$

Theorem 0.4 *Suppose* V *satisfies* Assumption 0.1. *Let* $1 \leq r \leq \dfrac{k}{k-1}$ *and let* $f(x,u)$ *satisfy the following conditions:*

(1) There exists a compact subset $K \subset \mathbb{R}$ *such that* $f(x,u) = 0$ *for* $x \notin K$*;*

(2) For a positive constant C,

$$\begin{array}{ll} |f(x,u)| \leq C|u|^r, & x \in \mathbb{R},\ u \in \mathbb{C}, \\ |f(x,u) - f(x,v)| \leq C|u-v|(|u|^{r-1} + |v|^{r-1}), & u, v \in \mathbb{C}. \end{array} \tag{0.5}$$

Then for any $u_0 \in L^2(\mathbb{R})$, (0.4) *is locally well-posed in* $L^{2r}_{\text{loc}}(\mathbb{R}_t \times \mathbb{R}_x) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x))$, *viz. there exists* $\delta > 0$ *such that* (0.4) *admits a unique solution* $u(t,x)$ *for* $-\delta < t < \delta$ *such that* $u \in L^{2r}_{loc}((-\delta,\delta) \times \mathbb{R}_x) \cap C((-\delta,\delta), L^2(\mathbb{R}_x))$ *and* $L^2(\mathbb{R}) \ni u_0 \mapsto L^{2r}_{\text{loc}}((-\delta,\delta) \times \mathbb{R}_x) \cap C((-\delta,\delta), L^2(\mathbb{R}_x))$ *is continuous. If* f *further satisfies*

$$f(x,u)\overline{u} \text{ is real for } x \in \mathbb{R}, \quad u \in \mathbb{C}, \tag{0.6}$$

then (0.4) *is globally well-posed in* $L^{2r}_{\text{loc}}(\mathbb{R}_t \times \mathbb{R}_x) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x))$, *viz. the preceding statement holds with* $\mathbb{R}$ *in place of* $(-\delta,\delta)$.

Theorem 0.5 *Suppose* V *satisfies* Assumption 0.1. *Let* $1 \leq r < \dfrac{2k}{2k-1}$ *and et* $f(x,u)$ *satisfy the following conditions with some constant* $c > 0$ *and* $\phi(x) \in L^{\frac{4}{2-r}}(\mathbb{R})$*:*

$$\begin{array}{ll} |f(x,u)| \leq c|\phi(x)||u|^r, & x \in \mathbb{R}, \quad u \in \mathbb{C} \\ |f(x,u) - f(x,v)| \leq c|\phi(x)||u-v|(|u|^{r-1} + |v|^{r-1}), & x \in \mathbb{R}, \quad u, v \in \mathbb{C} \end{array} \tag{0.7}$$

Then for any $u_0 \in L^2(\mathbb{R})$, (0.4) *is locally well-posed in* $L^4(\mathbb{R}_x; L^{2r}_{loc}(\mathbb{R}_t)) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x))$*. If* f *further satisfies* (0.6) *of the previous theorem, then* (0.4) *is globally well-posed in* $L^4(\mathbb{R}_x; L^{2r}_{loc}(\mathbb{R}_t)) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x))$.

IV. Main Estimates

We denote by $\psi(x,E)$ the L^2 normalized eigenfunctions of the operator $H = -\Delta + V(x)$:

$$-\psi''(x,E) + V(x)\psi(x,E) = E\psi(x,E). \tag{0.8}$$

We estimates the L^p-norm of $\psi(x,E)$ as the eigenvalue E tends to infinity and prove the following

Proposition 0.6 *For $1 \leq p < \infty$ we have*

$$\|\psi(x,E)\|_{L^p} \leq \begin{cases} CE^{-\frac{1}{k}(\frac{1}{2}-\frac{1}{p})}, & \text{if} \quad 1 \leq p < 4; \\ CE^{-\frac{1}{4k}}(\log E)^{\frac{1}{4}}, & \text{if} \quad p = 4; \\ C_p E^{\frac{k-4}{12k}-\frac{k-1}{3pk}}, & \text{if} \quad p > 4, \end{cases}$$

where the constant C_p for $p > 4$ may be taken uniformly outside a neighbourhood of 4.