Young Measure Solutions for a Class of Forward-Backward Diffusion Equations

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Abstract

We are concerned with the evolutionary equations of the form

$$\frac{\partial u}{\partial t} = \mathrm{div}\vec{\Psi}(\nabla u), \tag{1}$$

where $\vec{\Psi}(\xi)$ is the gradient of a smooth scalar function $\Phi(\xi)$ defined on $\mathbb{R}^N$, which is assumed without convexity but with the following growth conditions only

$$(\lambda|\xi|^p - 1)^+ \leq \Phi(\xi) \leq \Lambda|\xi|^p + 1, \qquad \forall \xi \in \mathbb{R}^N, \tag{2}$$

$$|\vec{\Psi}(\xi)| \leq \Lambda|\xi|^{p-1}, \qquad \forall \xi \in \mathbb{R}^N, \tag{3}$$

where p, λ, Λ are constants with $1 \leq p \leq 2$, $0 < \lambda \leq \Lambda$, and $s^+ = \max\{s, 0\}$.

Although the well-known evolutionary p-Laplacian equation with $1 < p \leq 2$ is a special case of the equations with the above conditions, we would prefer to make our main efforts to two other quite different kinds of such equations for the extremal case $p = 1$. The first typical example is

$$\frac{\partial u}{\partial t} = \mathrm{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \tag{4}$$

which can be regarded as a model for phase transition. Another typical example is a perturbated Perona-Malik's model for image processing, namely,

$$\frac{\partial u}{\partial t} = \mu\mathrm{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + \mathrm{div}\frac{\nabla u}{1 + |\nabla u|^2}, \quad (\mu > 0) \tag{5}$$

which is of the forward and backward type. Owing to the background, in both the above mentioned two examples, the solutions may be expected to be discontinuous or

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with unbounded spatial gradients. From mathematical point of view, the equation with $1 < p \leq 2$ can not be taken as the models having discontinuous solutions. Based on this consideration, we are much interested in the extremal case $p = 1$, and treating the case $1 < p \leq 2$ as the regular approximation.

First of all, we should first choose a reasonable solutions space to discuss the solvability. It is natural to choose BV, the class of all functions with bounded variation, as the desired solution space. However, for a function u in the BV class, what we can only know is that the first order derivatives of u are regular measures with bounded variation. So, we must give a reasonable explanation for the composition $\vec{\Psi}(\nabla u)$. For this purpose, we adopt the method inspired by the idea from a recent work of Demoulini, who treated the case $p = 2$ by means of Young measures. The compositions and hence the solutions are then defined by means of Young measures. We first prove the the existence for the regular case, and consequently, approximate the extreme case by the method of regularization. By means of some uniform estimates and some techniques for weak convergence, we establish the existence of Young measure solutions with bounded variation. Further, we discuss the properties of Young measure solutions, such as the uniqueness, stability, and the asymptotic behaviour.