Intermediate Asymptotics of Strong Solutions of the Thin Film Equation

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In this lecture we analyze the asymptotic behavior of the one–dimensional fourth–order nonlinear degenerate diffusion equation

$$\frac{\partial u}{\partial t} = -(uu_{xxx})_x, \quad u(x, t = 0) = u_0(x) \geq 0, \qquad (x \in \mathbb{R}, t > 0), \tag{1}$$

This equation, derived from a lubrication approximation, models the surface tension dominated motion of thin viscous films and spreading droplets. Equation (1) is a particular case of the thin film equation

$$\frac{\partial u}{\partial t} = -(|u|^n u_{xxx})_x, \qquad (x \in \mathbb{R}, t > 0), \tag{2}$$

where $n > 0$. Compactly supported nonnegative source type solutions to (2) exist for all $0 < n < 3$. Recent numerical studies [1] showed a rapid convergence of the solution onto the similarity solution before the merging of support. Here we will give a rigorous proof of this convergence. The main result we will prove is the following [2].

Theorem *The intermediate asymptotics in L^1 for strong positive solutions of the thin film equation* (1) *are given by the unique strong source type solution of the equation with the same mass. Moreover, an explicit and universal algebraic rate of convergence can be obtained.*

The result follows from techniques borrowed from kinetic theory of rarefied gases, essentially based on the study of the time decay of the entropy. Instead of working on (1) directly, one considers the asymptotic decay towards its equilibrium state of solutions to the (nonlinear) equation

$$\frac{\partial v}{\partial t} = (xv - vv_{xxx})_x, \quad v(x, t = 0) = u_0(x) \geq 0, \qquad (x \in \mathbb{R}, t > 0). \tag{3}$$

References

[1] A. L. Bertozzi, *The mathematics of moving contact lines in thin liquid films*, Notices of the AMS, June–July 1998, 689–697 (1998).

[2] J. A. Carrillo, G. Toscani, *Intermediate asymptotics for strong solutions of the thin film equation*, Preprint.

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