

ON THE EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTIONS FOR A SEMILINEAR PARABOLIC ABSTRACT EVOLUTION EQUATION

by

Francisco Periago

Dpto. Matemática Aplicada y Estadística. UPCT
C/Paseo Alfonso XIII, 30203 Cartagena, Spain

and

Bernd Straub

School of Mathematics. The University of NSW
2052 Sydney, NSW, Australia

Abstract. Sectorial operators, that is, linear operators A defined in Banach spaces, whose spectrum lies in a sector $S_\varphi = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \varphi\}$ for some $0 < \varphi < \frac{\pi}{2}$, and whose resolvent satisfies an estimate

$$\|(z - A)^{-1}\| \leq M |z|^{-1} \quad \text{for all } z \in \mathbb{C} \setminus S_\varphi; \quad (1)$$

have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. From an abstract point of view it is of particular interest to construct functional calculi for this class of operators and wide classes of holomorphic functions. Then complex powers of an operator A and semigroups associated with these powers are defined through the functional calculi. Finally, these abstract results can be applied to linear and nonlinear partial differential equations, mainly in the parabolic case, that is, when $0 < \varphi < \frac{\pi}{2}$ (see [2, 7, 11] for material regarding functional calculi, and [1, 5, 6, 8, 10] for the applications).

Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [3] and [6, Chapter 3]). However, if we look at spaces of more regular functions such as the spaces of Hölder continuous functions, we find that these elliptic operators do no longer satisfy the estimate (1) and therefore are not sectorial, as was pointed out by W. von Wahl in the case of the Laplacian (see [12] or [6, Example 3.1.33]).

Nevertheless, for these operators estimates such as

$$\|(z - A)^{-1}\| \leq M |z|^{-\alpha} \quad \text{for all } z \in \mathbb{C} \setminus S_\varphi \quad (2)$$

and some $\frac{1}{2} < \alpha < 1$, can be obtained.

In the talk we will consider the semilinear abstract Cauchy problem

$$(SLP) \quad \begin{cases} u'(t) + Au(t) = f(t; u(t)); & 0 < t < T \\ u(0) = u_0; \end{cases}$$

where $f :]0; T[\times X \rightarrow X$ and $u_0 \in X$ are given, and A is an operator which satisfies (2) for some $0 < \alpha < \frac{1}{2}$. As in the case of sectorial operators we will see that (SLP) has a unique mild solution, that is, a solution of the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s; u(s)) ds \quad (IE)$$

where $T(t)$ is the analytic semigroup of growth order $\alpha + 1$ associated with A : The latest basically means that the semigroup behaves as $O(t^{-(\alpha+1)})$ as time goes to zero. Under suitable regularity

conditions on the initial data we prove that this solution is classical. To prove this result we first investigate the inhomogeneous linear Cauchy problem associated with A and the semigroups generated, in some sense, by A : All of this is based on a functional calculus that we construct for A and which is based on the ideas underlying the McIntosh functional calculus for sectorial operators (see [2, 4, 7]).

Finally we give some examples of $2m_j$ order elliptic operators in the space of Hölder continuous functions for which our abstract result applies.

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