On the relativistic Euler equation

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This is a joint work with Cheng-Hsiung Hsu (NCU, TW) and Song-Sun Lin (NCTU, TW). We study the Cauchy problem to the relativistic Euler equation

\[
\frac{\partial}{\partial t} \left( \frac{\rho + P u^2}{1 - u^2/c^2} \right) + \frac{\partial}{\partial x} \left( \frac{\rho (\rho + P)/c^2}{1 - u^2/c^2} \right) = 0, \\

\frac{\partial}{\partial t} \left( \frac{\rho + P u^2}{c^4} \right) \frac{1}{1 - u^2/c^2} + \frac{\partial}{\partial x} \left( \frac{\rho u^2}{c^2} + \frac{P}{c^2} \right) = 0, \\
\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x). \tag{1}
\]

Here \( P \) is a given function of \( \rho \) and \( c \) is a positive constant, the speed of light. The first mathematical investigation of this problem was done by J. Smoller and B. Temple [1] 1993. They assume that \( P = \sigma^2 \rho \), \( \sigma \) being a positive constant \( < c \) and

\[
T.V. \log \rho_0 + T.V. \log \frac{c + u_0}{c - u_0} < \infty
\]

to show the existence of global weak solutions to (1)(2). The scheme is the Glimm’s and the discussion on the large initial data follows T. Nishida [2] 1968. We are interested in a more realistic relation between \( P \) and \( \rho \). Keeping in mind the equation of states for neutron stars

\[
P = Kc^5 f(y), \quad \rho = Kc^3 g(y), \\
f(y) = \int_0^y \frac{q^4}{\sqrt{1 + q^2}} dq, \quad g(y) = 3 \int_0^y q^2 \sqrt{1 + q^2} dq,
\]

we assume

(A): \( P > 0, 0 < dP/d\rho < c^2, 0 < d^2P/d\rho^2 \) for \( \rho > 0 \), and

\[
P = A \rho^\gamma (1 + P_1(\rho^{-1}/c^2)) \quad \text{as} \quad \rho \to 0,
\]

where \( A \) and \( \gamma \) are positive constants, \( \gamma = 1 + \frac{2}{2N+1} \), \( N \) being a positive integer and \( P_1(X) \) is a convergent power series such that \( P_1(0) = 0 \). Our main result is: For any \( M_0 \) there is a positive number \( \epsilon_0 = \epsilon_0(M_0) \) such that if

\[
0 \leq \rho_0(x) \leq M_0, \quad \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \leq M_0
\]
and if $1/c^2 \leq \epsilon_0$, then there is a global weak solution to (1)(2). As a corollary we have: There is $\epsilon_1 > 0$ such that if

$$0 \leq \rho_0(x) \leq \epsilon_1 c \frac{2}{c - u_0(x)}, \quad \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \leq \epsilon_1 c,$$

then there is a weak solution to (1)(2). Approximate solutions are constructed by the Lax-Friedrichs or Godunov scheme. The Riemann invariants are $w = x + y, z = x - y$, where

$$x = \frac{c}{2} \log \frac{c + u}{c - u}, \quad y = \int_0^\rho \sqrt{\frac{P'}{\rho + P'/c^2}} d\rho.$$

In order to show the convergence of approximate solutions we must find many entropies, which are solutions to the "relativistic Euler-Poisson-Darboux equation"

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial y^2} + A(x, y) \frac{\partial \eta}{\partial y} + B(x, y) \frac{\partial \eta}{\partial x} = 0, \quad (3)$$

where

$$A = \frac{1}{\sqrt{\rho}} \left(1 - \frac{P'}{c^2} \rho + \frac{P'/c^2}{2P'} P'' \right) \frac{1}{1 - P''/c^4},$$

$$B = -\frac{2u/c^2}{1 - P''/c^4} \left(1 - \frac{P'}{c^2} - \frac{\rho + P'/c^2}{2P'} P'' \right).$$

We constructed the "generalized Darboux formula"

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi,$$

which gives a solution of (3) for any smooth $\phi$. Here $K(x, y, \xi)$ is of class $C^{N+2}$ in $|x| < \infty, 0 \leq y, |x - \xi| \leq y$ and

$$K(x, y, \xi) = (y^2 - (x - \xi)^2)^N \left(1 + O(y/c^2)\right).$$
