On the relativistic Euler equation

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This is a joint work with Cheng-Hsiung Hsu (NCU, TW) and Song-Sun Lin (NCTU, TW). We study the Cauchy problem to the relativistic Euler equation

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{(\rho + P/c^2)u}{1-u^2/c^2} &= 0, \\ \frac{\partial}{\partial t}\frac{(\rho + P/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{\rho u^2 + P}{1-u^2/c^2} &= 0, \end{aligned} \qquad (1)$$

$$\rho|_{t=0} = \rho_0(x) \quad , \qquad u|_{t=0} = u_0(x). \qquad (2)$$

Here P is a given function of ρ and c is a positive constant, the speed of light. The first mathematical investigation of this problem was done by J. Smoller and B. Temple [1] 1993. They assume that $P = \sigma^2\rho$, σ being a positive constant $< c$ and

$$T.V. \log\rho_0 + T.V. \log\frac{c+u_0}{c-u_0} < \infty$$

to show the existence of global weak solutions to (1)(2). The scheme is the Glimm's and the discussion on the large initial data follows T. Nishida [2] 1968. We are interested in a more realistic relation between P and ρ. Keeping in mind the equation of states for neutron stars

$$\begin{aligned} P &= Kc^5 f(y), \qquad \rho = Kc^3 g(y), \\ f(y) &= \int_0^y \frac{q^4}{\sqrt{1+q^2}}dq, \qquad g(y) = 3\int_0^y q^2\sqrt{1+q^2}dq, \end{aligned}$$

we assume

(A): $P > 0, 0 < dP/d\rho < c^2, 0 < d^2P/d\rho^2$ for $\rho > 0$, and

$$P = A\rho^\gamma(1 + P_1(\rho^{\gamma-1}/c^2)) \qquad as \qquad \rho \to 0,$$

where A and γ are positive constants, $\gamma = 1 + \frac{2}{2N+1}$, N being a positive integer and $P_1(X)$ is a convergent power series such that $P_1(0) = 0$. Our main result is: *For any M_0 there is a positive number $\epsilon_0 = \epsilon_0(M_0)$ such that if*

$$0 \leq \rho_0(x) \leq M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq M_0$$

and if $1/c^2 \leq \epsilon_0$, then there is a global weak solution to (1)(2). As a corollary we have: *There is $\epsilon_1 > 0$ such that if*

$$0 \leq \rho_0(x) \leq \epsilon_1 c^{\frac{2}{\gamma-1}}, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq \epsilon_1 c,$$

then there is a weak solution to (1)(2). Approximate solutions are constructed by the Lax-Friedrichs or Godunov scheme. The Riemann invariants are $w = x+y, z = x-y$, where

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

In order to show the convergence of approximate solutions we must find many entropies, which are solutions to the "relativistic Euler-Poisson-Darboux equation"

$$\frac{\partial^2\eta}{\partial x^2} - \frac{\partial^2\eta}{\partial y^2} + A(x,y)\frac{\partial\eta}{\partial y} + B(x,y)\frac{\partial\eta}{\partial x} = 0, \tag{3}$$

where

$$\begin{aligned} A &= \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P'')\frac{1 + P'u^2/c^4}{1 - P'u^2/c^4}, \\ B &= -\frac{2u/c^2}{1 - P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P''). \end{aligned}$$

We constructed the "generalized Darboux formula"

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi,$$

which gives a solution of (3) for any smooth ϕ. Here $K(x,y,\xi)$ is of class C^{N+2} in $|x| < \infty, 0 \leq y, |x-\xi| \leq y$ and

$$K(x,y,\xi) = (y^2 - (x-\xi)^2)^N(1 + O(y/c^2)).$$

Then the proof is done along the discussions of R. DiPerna [3] 1983 and G.-Q. Chen et al [4] 1985-86.

[1] Commun. Math. Phys., 156(1993), 67-99.
[2] Proc. Japan Acad., 44(1968),642-646.
[3] Arch. Rat. Mech. Anal., 82(1983), 27-70.
[4] Acta Math. Scientica, 5(1985), 415-432, 5(1985), 433-472, 6(1986), 75-120.