The entropy/entropy production method for the large time behavior of Fokker-Planck type

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In this talk we are concerned with the large-time behavior of the Cauchy problem for linear and nonlinear Fokker–Planck type equations (advection-diffusion equations). These problems appear as homogeneous versions of kinetic Fokker–Planck models, e.g. In particular, we use the entropy method to analyze the rate of convergence to the equilibrium. Initially, we consider the IVP for the *Fokker-Planck type equation*

$$\rho_t = \mathrm{div}(\nabla\rho + \rho\nabla A), \quad x \in \mathbb{R},\, t > 0; \quad \rho(t=0) = \rho_I \in L^1_+(\mathbb{R}^n)$$

with a given potential $A = A(x)$ such that the steady state $\rho_\infty = e^{-A} \in L^1(\mathbb{R})$.

For a wide class of problems the solution converges (in a weighted L^2-space) exponentially to the unique steady state, which is due to the spectral gap of the Hamiltonian that governs the evolution of the symmetrized problem. To extend this result to more general initial data we analyze the time decay of the *relative entropy* $e(t) = e_\psi(\rho(t)|\rho_\infty) = \int_{\mathbb{R}} \psi(\frac{\rho(t)}{\rho_\infty})\, \rho_\infty(dx)$, generated by the convex function $\psi(\sigma)$, $\sigma > 0$ with $\psi(1) = \psi'(1) = 0$. The idea of our analysis is to derive a differential inequality for the entropy production $I(t) = I_\psi(\rho(t)|\rho_\infty) = \frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) \le 0$ and the entropy production rate $I'(t)$. For uniformly convex $A(x)$, i.e. $\left(\frac{\partial^2 A(x)}{\partial x^2}\right) \ge \lambda \mathbf{I}$, $x \in \mathbb{R}$, we get

$$\frac{d}{dt} I(t) = -2\lambda I(t) + r_\psi(\rho(t)), \tag{1}$$

with a non-negative remainder $r_\psi(\rho(t))$. This shows the exponential decay of $I(t)$. An integration in t then gives

$$I(t) = e'(t) \le -2\lambda e(t), \tag{2}$$

and the relative entropy converges to 0 exponentially with rate -2λ.

(2) is the entropy version of a convex Sobolev inequality. For the physical entropy ($\psi(\sigma) = \sigma \ln\sigma - \sigma + 1$) and Gaussian steady states $\rho_\infty = M_a$ (with standard deviation a) (2) can be transformed to the well-known *Gross logarithmic Sobolev inequality* (L. Gross, *Lecture Notes in Mathematics*, **1563**, 1993):

$$\int_{\mathbb{R}} f^2 \ln\left(\frac{f^2}{\|f\|^2_{L^2(dM_a)}}\right) M_a(dx) \le 2a \int_{\mathbb{R}} |\nabla f|^2 M_a(dx), \quad \forall f \in L^2(dM_a).$$

From the precise form of the remainder r_ψ in (1) one can deduce sharpness conditions on ρ, ρ_∞, and ψ such that (2) becomes an equality.

In this talk we shall also discuss extensions of the entropy method to non-linear models, non-symmetric diffusions, and non-logarithmic entropy functionals.

References

[1] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter. On logarithmic Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. to appear in Comm. PDE, (2000).

[2] A. Arnold, E. Carlen. A generalized Bakry-Emery condition for non-symetric diffusions, to appear in the Proceedings of Equadiff99, Berlin 1999, World Scientific Publishing (2000).

[3] A. Arnold, J. Dolbeault. Refined Convex Sobolev Inequalities. preprint, (2000).