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## Scientific Computations Related to the Riemann Hypothesis

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It is well known that the Riemann Hypothesis is equivalent to the statement that all zeros of the Riemann function  $\Xi(x)$  are real. This function can be expressed as the integral

$$\Xi\left(\frac{x}{2}\right)/8 = \int_0^{\infty} \Phi(u) \cos(xu) du \quad (x \in \mathbb{C}),$$

where

$$\Phi(u) := \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}) \quad (0 \leq u < \infty).$$

Pólya considered the more general trigonometric integral

$$H_t(x) := \int_0^{\infty} e^{tu^2} \Phi(u) \cos(xu) du \quad (t \in \mathbb{R}; x \in \mathbb{C}),$$

where we see that  $H_o$  and the  $\Xi$ -function are related through

$$H_o(x) = \Xi\left(\frac{x}{2}\right)/8,$$

so that the Riemann Hypothesis is also equivalent to the statement that all zeros of  $H_o$  are real.

Based on the work of de Bruijn(1950), C. Newman(1976) showed that there is a real constant  $\Lambda$ , with  $-\infty < \Lambda \leq 1/2$ , such that  $H_t$  has only real zeros if and only if  $t \geq \Lambda$ . This constant  $\Lambda$  is now called the **de Bruijn-Newman constant** in the literature and it follows that

the Riemann Hypothesis is true if and only if  $\Lambda \leq 0$ .

Lower bounds for  $\Lambda$ , such as

$$-5.895 \cdot 10^{-9} < \Lambda \quad \text{and} \quad -2.7 \cdot 10^{-9} < \Lambda,$$

have quite recently been determined, and we show how analysis techniques, combined with known numerical values for some zeros of  $\Xi(x)$ , produce these lower bounds above. (Curiously, no upper bound for  $\Lambda$ , better than de Bruijn's original  $\Lambda \leq 1/2$ , has been established.)