Local Existence with Minimal Regularity for Semilinear Wave Equations

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In this talk, we consider the following systems of semilinear wave equations in n space dimensions:

$$\Box\phi = F(\partial\phi) \tag{1}$$

where $\Box = \partial_t^2 - \triangle$ is the wave operator, $\triangle = \sum_{i=1}^n \partial_{x_i}^2$ stands for the Laplacian, $\phi = (\phi^1, \phi^2, \cdots, \phi^m)$, $F = (F^1, F^2, \cdots, F^m)$, $\partial = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$. Moreover, F^i are quadratic functions of $\partial\phi$. We prescribe the following Cauchy data at time $t = 0$:

$$\phi(0, x) = f(x), \quad \partial_t\phi(0, x) = g(x) \tag{2}$$

where $f \in H^s, g \in H^{s-1}$.

We want to determine the minimal value of s such that the Cauchy problem (1)(2) is locally well posed in H^s. The classical local existence theorem requires $s > \frac{n}{2} + 1$ while the scaling limit is $s > \frac{n}{2}$. The correct one turns out to be

$$s > max\{\frac{n}{2}, \frac{n+5}{4}\}. \tag{3}$$

The counter examples which show that (3) can not be improved is due to Lindblad. The positive result is due to Sideris et al (when n=3) and Tartaru (when $n \geq 5$). The proof in the case $n = 2, 4$ was recently supplemented by the author.

when F satisfies the so called "null condition", that is, F^i are all linear combinations of the following null forms:

$$Q_0(\phi, \psi) = \partial_t\phi\partial_t\psi - \sum_{i=1}^n \partial_{x_1}\phi\partial_{x_1}\psi \tag{4}$$

$$Q_{\alpha\beta}(\phi, \psi) = \partial_{x_\alpha}\phi\partial_{x_\beta}\psi - \partial_{x_\beta}\phi\partial_{x_\alpha}\psi \quad (\alpha, \beta = 0, 1, \cdots, n). \tag{5}$$

Then the local well posedness result can be improved. For the first null form, one can prove local well posedness up to the scaling limit $s > \frac{n}{2}$, this is due to Klainerman and Machedon (when n=3,4), Klainerman and Selberg (when n=2), Keel and Tao (when n=1). For the second null form , the correct exponent is

$$s > max\{\frac{n}{2}, \frac{n+3}{4}\}. \tag{6}$$

This is due to Klainerman and Machedon (when n=3,4) and the author (when n=1,2).