Paley-Wiener Theorem for Dunkl Transform

VU KIM TUAN
Department of Mathematics and Computer Science
Kuwait University, Kuwait
E-mail: vu@mcs.sci.kuniv.edu.kw

For $\alpha \in R^n \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset R^n$ orthogonal to α. A finite set $\mathcal{R} \subset R^n \backslash \{0\}$ is called a root system if $\mathcal{R} \cap R.\alpha = \{\pm\alpha\}$ and $\sigma_\alpha \mathcal{R} = \mathcal{R}$ for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$ the reflections σ_α generate a finite group $W \subset O(n)$. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in R^n \backslash \cup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R} : (\alpha, \beta) > 0\}$. We assume the root system $\mathcal{R}$ is normalized in the sense that $|\alpha| = \sqrt{2}$ for all $\alpha \in \mathcal{R}$. A function $k : \mathcal{R} \to C$ on a root system $\mathcal{R}$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. Denotes the number of conjugacy classes of reflections by m. Let $K = C^m$.

The Dunkl operators $T_\zeta, \zeta \in R^n$, on R^n associated with the finite reflection group W and multiplicity function k are given by

$$T_\zeta f(x) = \partial_\zeta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_i \, . \, \frac{f(x) - f(\sigma_\alpha)}{< \alpha, x >}.$$

Consider the system

$$T_\zeta(k) f = (\lambda, \zeta) f. \tag{1}$$

There exists an open set $K^{reg} \subset K$ invariant under complex conjugation and containing $\{k \in K | \Re(k) \geq 0\}$ such that the solution space of (1) is 1-dimensional for all $k \in K^{reg}$ and $\lambda \in C^n$. This solution space contains a unique function $Exp_G(\lambda, k, .)$ such that $Exp_G(\lambda, k, 0) = 1$.

Let $\Re(k) \geq 0$. Put

$$w_k(x) = \prod_{\alpha \in \mathcal{R}_+} |(\alpha, x)|^{2k_\alpha}.$$

The Dunkl transforms are defined as follows

$$(D_k f)(\lambda) = \frac{1}{c_k} \int_{R^n} f(x) Exp_G(-i\lambda, k, x) w_k(x) dx,$$

$$(E_k f)(x) = \frac{1}{c_k} \int_{R^n} f(\lambda) Exp_G(i\lambda, k, x) w_k(\lambda) d\lambda.$$

The constant c_k is known as a Mehta-type integral. The Plancherel theorem for the Dunkl transforms says that D_k and E_k are unitary operators on $L_2(R^n, |w_k(x)|\, dx)$, and they are the inverses of each other.

In this talk we establish a Paley-Wiener-type theorem for the Dunkl transforms of functions with compact support. The characterization is formulated on R^n without passing to complexification.