# Finite Element Approximation of Conformal Mappings and Its Applications to a Free Boundary Problem

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In this lecture we discuss finite element approximation of conformal mappings. Then we apply the obtained scheme to compute solutions of a classical free boundary problem — the **dam** (or filtration) **problem**. Let $B \subset \mathbb{R}^2$ be the unit disk and $\gamma \subset \mathbb{R}^2$ a given closed Jordan curve. Conformal mappings $x(u,v)$, $((u,v) \in B)$ are defined on $B$ as solutions to the following partial differential equations:

$$\Delta x = 0 \quad \text{in } B,$$
$$|x_u|^2 - |x_v|^2 = (x_u, x_v) = 0 \quad \text{in } B,$$
$$x(\partial B) = \gamma \text{ and } x|_{\partial B} : \partial B \to \gamma \text{ is homeomorphic.}$$

Here, $x_u$, $x_v$ are the partial derivatives with respect to $u$, $v$, respectively, and $(\cdot,\cdot)$, $|\cdot|$ are the inner product in $\mathbb{R}^2$ and the 2-norm defined by that.

For the conformal mappings, the following variational principle has been known. Let the subset $X_\gamma$ of mappings on $B$ be defined by

$$X_\gamma := \left\{\psi \in C(\overline{B};\mathbb{R}^2) \cap H(B;\mathbb{R}^2) \;\middle|\; \psi(\partial B) = \gamma,\ \psi|_{\partial B} \text{ is monotone}\right\},$$

where $\psi|_{\partial B}$ being monotone means that $(\psi|_{\partial B})^{-1}(p)$ is connected for all $p \in \gamma$. Then, a map $x \in X_\gamma$ is a solution of the above system if and only if $x \in X_\gamma$ is a stationary point of the energy functional $D(x)$ in $X_\gamma$. Here, $D$ is the Dirichlet integral defined by

$$D(\psi) := \int_B |\nabla\psi|^2 \mathrm{d}u\mathrm{d}v.$$

To find a conformal mapping defined on $B$, therefore, we just need to find a stationary point of the Dirichlet integral in $X_\gamma$.

Here, we approximate the Dirichlet integral by piecewise linear finite elements. It has been shown that, if the Jordan curve is rectifiable, the finite element conformal mappings converge to the exact conformal mappings when triangulation is getting finer.

We then try to apply this scheme to solve the dam (or filtration) problem [1], [2], a typical and classical free boundary problem. A detail of formulation and numerical examples will be given at the lecture.

## References

[1]. A. Friedman. *Variational Principles and Free-Boundary Problems*, Wiley, 1982.

[2]. D. Kinderlehrer, G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*, Academic, 1980.