Solutions of Fractional Multi-order Differntial Equations Using Poisson-type Transform as Transmutation Operator

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We consider a wide class of ordinary differential equations of fractional multi-orders $(1/\rho_1, 1/\rho_2, \ldots, 1/\rho_m)$, depending on arbitrary parameters $\rho_i > 0, \mu_i \in \mathbb{R}, i = 1, \cdots, m$. Denoting the "differentiation" operators by $\mathcal{D} = D_{(\rho_i),(\mu_i)}$, and by $\mathcal{L} = L_{(\rho_i),(\mu_i)}$-the inverse integrations, we first observe that $\mathcal{D}$ and $\mathcal{L}$ can be considered as operators of the generalized fractional calculus, respectively-as generalized fractional "derivatives" and "integrals". The solution of the homogeneous ODE of this kind,

$$\mathcal{D}y(z) = -y(z), y(0) = 1, y'(0) = \cdots = y^{(m-1)}(0) = 0, \quad 0 < |z| < \infty,$$

is the recently introduced "multi-index Mittag-Leffler function".

We find a Poisson-type integral transformation $\mathcal{P}$ (generalizing the classical Poisson integral formula) that maps the $\cos_m$-function into the multi-index Mittag-Leffler function, and also transforms the simpler differentiation and integration operators of integer order $m > 1 : D^m = (\frac{d}{dz})^m$ and j^m (the m-fold integration) into the operators $\mathcal{D}$ and $\mathcal{L}$. Thus, from the known solution of the volterra type integral equation with the m-fold integration l^m, via $\mathcal{P}$-as a transformation (transmutation) operator, we find the corresponding solution of the integral equation $y(z) - \lambda\mathcal{L}(z) = f(z)$. Then, the solution of an initial value problem for the fractional multi-order ODE: $\mathcal{D}y(z) - \lambda y(z) = f(z)$ comes out, in an explicit form, as a series of integrals involving Fox's H-functions. For each particularly chosen R.H.S. function $f(z)$, such a solution can be explicitly evaluated as an H-function. Special cases of the equations considered here, lead to solutions in terms of the Mittag-Leffler, Bessel, Struve, Lommel and hyper-Bessel functions, and some other known generalized hypergeometric functions.

Keywords: fractional order differential and integral equations, operators of generalized fractional calculus, Mittag-Leffler function, Fox's H-funcation, method of transmutations.