Anisotropic dissipative effects on the buoyancy instability with background heat flux

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The linear buoyancy instability in magnetized plasmas is investigated in the presence of anisotropic resistivity and viscosity by taking into account the background heat flux. The magnetic field is assumed to be homogeneous and has both horizontal and vertical components. The heat is primarily transported along the magnetic force lines when the gyro radius is much less than the mean collision free path. The Hall term is examined first and shows a damping effect on the magnetothermal instability. The heat-flux-driven buoyancy instability (HBI) is then investigated by taking into account the parallel resistivity (PR), cross-field resistivity (CR), and the anisotropic viscosity. The general dispersion relation (DR) is derived and discussed in several special cases. We show that only the CR and viscosity exert effects on the DR in the first case. The critical condition for the occurrence of HBI is modified by the CR coupled with the viscosity and the value of the instability growth rate is diminished by them. The effects due to the PR (resp. viscosity) on the HBI are examined next. The PR (resp. viscosity) is shown to alter not only the growth rate but also the instability criterion. There exists an unstable mode when the temperature decreases in the direction of gravity while this case is proven to be magnetothermally stable in the ideal magnetohydrodynamic limit. A new unstable mode is solely induced by the presence of PR (resp. viscosity). When the PR and CR are both taken into account, the resistivity is shown to induce a damping mode rather than an instability. Finally, considering the PR and viscosity simultaneously, it is found that a new unstable mode is excited when the PR is not equal to the viscosity, or else, dissipation effects do not alter the instability criterion and just cut down the growth rate.


I. INTRODUCTION

When the mean free path between particle collisions is much greater than the particle Larmor radius in a magnetized plasma, i.e., \( \omega_c \tau \gg 1 \), where \( \omega_c \) is the gyro frequency and \( \tau \) is the mean collision time, the heat is restricted to being transported primarily along the magnetic force lines.\(^1\) Under this circumstance, it is sufficient to adopt the Braginskii’s magnetohydrodynamic (MHD) equations to describe the plasmas. That is, the anisotropic transport terms must be taken into account in the MHD equations. For instance, both the heat flux and resistivity have two different components, parallel (with respect to the magnetic field) and cross-field projection, and viscosity is described by an anisotropic tensor.

An anisotropic thermally stratified plasma has been shown to be buoyantly unstable when the thermal temperature increases in the direction of gravity.\(^2,3\) This thermal convective instability is referred to as magnetothermal instability (MTI).\(^4\) MTI was initially investigated in 2000 by Balbus, who examined a simple case in which a horizontal magnetic field existed in a vertically stratified plasma in the absence of the background heat flux and found that the criterion for convective instability went from one of upwardly decreasing entropy to one of upwardly decreasing temperature.\(^5\) In a subsequent paper, Balbus investigated MTI in the differential rotational system and studied the combination between MTI and magnetorotational instability (MRI), but the heat flux in the background plasma was still not taken into account.\(^3\) Parrish and Stone used numerical methods to explore the nonlinear evolution and saturation of the MTI in two and three dimensions, respectively. They showed that the linear growth rates measured in the simulation agreed well with the weak-field dispersion relation.\(^4,5\) Quataert calculated the linear MTI in the presence of background heat flux by assuming that the magnetic field had both the vertical and horizontal components.\(^6\) It was found that the presence of a heat flux drove a buoyancy instability analogous to the MTI when the temperature decreases in the direction of gravity while this situation was magnetothermally stable according to Balbus’s analysis (see Ref. 2); the so-called heat-flux-driven buoyancy instability (HBI).\(^7\) Parrish et al. investigated the long-standing cooling flow effect on the HBI in galaxy clusters with three-dimensional MHD simulation of isolated clusters including radiative cooling.\(^7\) Dennis and Chandran examined the effect on the MTI of the cosmic rays and arbitrary magnetic field.\(^8\) Recently, Ren et al. investigated the density and magnetic field gradients effect,\(^9\) anisotropic resistivity, and viscosity effect on the MTI,\(^8,10\) and also discussed the combination between the MTI and MRI in the presence of aniso-
tropic dissipative effect. However, the background heat flux and anisotropic dissipation have not been considered simultaneously to analytically explore the buoyancy instability, and this is the objective of this study.

In the present work, we study the thermal convective instability in magnetized plasmas with background heat flux in the framework of anisotropic resistive and viscous MHD models. The magnetic field is assumed to be homogeneous with vertical and horizontal components in our calculation. By adopting the standard Wentzel–Kramers–Brillouin (WKB) approximation and working in the Boussinesq limit, the general DR is deduced with the help of different Ohm’s laws. By means of the Ohm’s law containing the isotropic resistivity and Hall term, we find that the Hall term exerts a stabilizing effect on the MTI. We then study the HBI by taking into account the parallel resistivity (PR) and cross-field resistivity (CR) as well as the anisotropic viscosity. In the first special case, only the CR and viscosity play effects on the DR, alter the instability criterion, and decrease the instability growth rate. The effect of PR on the DR, alter the instability criterion, and decrease the instability growth rate. The effect of PR (resp. viscosity) on the HBI are studied next. A new unstable mode is introduced by the presence of PR (resp. viscosity). Considering the PR and CR together, we find that the resistivity induces a damping mode; considering the PR and viscosity simultaneously, we find that a new unstable mode is induced when the PR is not equivalent to the viscosity. The present paper is organized as follows. The basic equations of a nonideal MHD model are presented in Sec. II. The Hall mode is discussed in Sec. III. The DR of HBI is derived in Sec. IV by virtue of anisotropic resistivity and viscosity. Effects due to the anisotropic resistivity and viscosity on the HBI are discussed in different cases in Sec. V. Finally, Sec. VI is the conclusion.

II. BASIC EQUATIONS AND ASSUMPTIONS

We start our derivation from the basic set of nonideal MHD equations with the addition of the heat flux, \( \dot{Q} \), and a gravitational field in the presence of anisotropic resistivity and viscosity, which are as follows.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}
\]

\[
\rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{J} \times \vec{B} + \rho \vec{g} + \nabla \cdot \vec{\Pi}, \tag{2}
\]

\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \tag{3}
\]

\[
\rho \frac{dS}{dt} = -\nabla \cdot \vec{Q} + \Pi_{ij} \partial_{ij} \vec{p} + \vec{J} \cdot (\vec{E} + \vec{v} \times \vec{B}). \tag{4}
\]

Here, \( \rho \) is the fluid mass density, \( \vec{v} \) is the fluid velocity, \( \partial / \partial t = \partial / \partial t + \vec{v} \cdot \nabla \) is the convective derivative, \( p \) is the isotropic thermal pressure, \( \vec{B} \) is the magnetic field, \( \vec{g} \) is the gravitational acceleration, \( \vec{\Pi} \) is the viscosity tensor, \( \vec{E} \) is the electric field, \( T \) is the temperature in the energy units, \( S = 3 p \ln(p \rho^{-1}) / 2 \rho \Gamma \) is the entropy per unit mass, and \( \vec{J} = \nabla \times \vec{B} / \mu_0 \) is the current density. The adiabatic index \( \Gamma \) is 5/3. The plasma is thermally stratified and placed in a uniform gravitational field along the vertical direction, \( \vec{g} = -g \hat{z} \).

All fluid variables depend only upon \( z \). \( \partial_t \) is denoted for \( \partial / \partial t \) and summation over repeated suffixes is assumed. The equilibrium magnetic field \( \vec{B}_0 = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \) is homogeneous and hence the equilibrium condition is \( d\rho_0 / dz = -\rho_0 \vec{g} \), where \( \rho_0 \) and \( p_0 \) are the equilibrium pressure and mass density, respectively. The heat flux contains both the Coulomb and radiative heat conductivity and reads

\[
\vec{Q} = -\chi_c \vec{b} \cdot (\nabla T) - \chi_R \nabla T, \tag{5}
\]

where \( \chi_c \) is the Spitzer Coulomb conductivity, \( \vec{b} = \vec{B} / B \) is a unit vector in the direction of the magnetic field, and \( \chi_R \) is the coefficient of isotropic conductivity. The equilibrium heat flux is given by

\[
\vec{Q}_0 = -\chi_c (c (b_x b_x \hat{x} + b_y b_y \hat{y} + b_z b_z \hat{z})) \frac{dT_0}{dz} - \chi_R \frac{dT_0}{dz} \hat{z}, \tag{6}
\]

where \( T_0 \) is the unperturbed temperature. Note that the initial heat flux in the equilibrium state should be steady, i.e., \( \nabla \cdot \vec{Q}_0 = 0 \), the temperature needs to vary linearly with height \( z \). The set of equations above is coupled with the Ohm’s law to determine the current density \( \vec{J} \) and then enclose the system. However, the Ohm’s law has different forms in different conditions. They will be discussed in detail below.

III. HALL MODE IN THE NONVISCID PLASMAS

In this section, we restrict ourselves to the nonviscous plasmas and follow the general Ohm’s law,

\[
\vec{E} = -\vec{v} \times \vec{B} + \eta_0 \vec{J} + \omega_e \tau_e \vec{J} \times \vec{B}. \tag{7}
\]

Here, \( \omega_e = e B / m_e \) is the electron gyro frequency, \( m_e \) is the electron mass, \( \tau_e \) is the electron-ion mean collision time, and \( \eta = m_e \nu_e / n e^2 \) is the resistivity, where \( \nu_e = \tau_e^{-1} \) is the electron-ion collision frequency, \( n \) is the plasma density, and \( e \) is the negative electron charge. The Ohmic term, \( \eta_0 \vec{J} \), is supposed to be isotropic. The last term on the right-hand side is the called Hall term. Perturbations are assumed to have the WKB space-time dependence, \( \propto \exp(-i \omega t + i k \cdot \vec{r}) \), where \( \omega \) is the wave frequency and \( \vec{k} = (k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \) is the wave vector. The WKB assumption requires \( k L \gg 1 \), where \( L \) is the local scale length of the system inhomogeneity and \( k = |\vec{k}| \) is the wave number. Furthermore, it is sufficient for us to work in the Boussinesq limit, that is, the fluctuation in pressure is much less than that in mass density in the energy equation, viz. \( \partial_t p / p_0 \ll \rho_1 / p_0 \), and in the perturbed mass conservation and momentum equations, the density variations can be neglected except when they are coupled to the gravitational term. The subscript 0 denotes the unperturbed quantity and 1 denotes the perturbed quantity hereinafter. Substituting the perturbed electric field from the Ohm’s law into the perturbed Eq. (3) yields the perturbed magnetic field.
Recall that $g$, in which we have used $\omega=\omega+i\eta k^2/\mu_0$, $v_H=\eta \omega_\perp \gamma$, $\epsilon$ is $\alpha/\mu_0$, and $\alpha$ is the angle between $\vec{k}$ and $\vec{B}_0$. The perturbed motion equation is

$$-i\omega p_0 \vec{v}_1 = -ik\left(p + \frac{\vec{B}_0 \cdot \vec{B}_0}{\mu_0}\right) + i\left(\vec{k} \cdot \vec{B}_0\right) \vec{B}_1 + p_1 \vec{g}.$$  \hspace{1cm} (9)

Adopting $\vec{k} \times [\text{Eq. (9)}]$ leads to

$$-i\omega p_0 \vec{k} \times \vec{v}_1 = i\left(\vec{k} \cdot \vec{B}_0\right) \vec{k} \times \vec{B}_1 + p_1 \vec{k} \times \vec{g}.$$  \hspace{1cm} (10)

Inserting $\vec{B}_1$ into the formula above, we obtain

$$\vec{v}_1 = \frac{v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{v}_1 - \frac{k^2 v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{v}_1 + \frac{p_1}{\mu_0} \vec{k} \times \vec{g},$$  \hspace{1cm} (11)

in which $\omega_\perp = kv_A \cos \alpha$ is the Alfvén frequency and $v_A = B_0/\sqrt{\rho_0 \mu_0}$ is the Alfvén speed. Timed by $\vec{k}$ again, the formula above gives

$$\vec{B}_1 = \frac{v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} - \frac{k^2 v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{B}_1 - \frac{p_1}{\mu_0} \vec{k} \times \vec{g}.$$  \hspace{1cm} (12)

In which we have used $\vec{k} \times (\vec{k} \times \vec{v}_1) = -k^2 \vec{v}_1$. Inserting the perturbed velocity $\vec{v}_1$ into Eq. (8) gives rise to

$$\vec{v}_1 = \frac{v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{v}_1 - \frac{k^2 v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{v}_1 + \frac{p_1}{\mu_0} \vec{k} \times \vec{g},$$  \hspace{1cm} (13)

Recall that $\vec{g} = -g \vec{z}$; we then have

$$\vec{B}_1 = \frac{v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{B}_1 - \frac{k^2 v_H \omega_\perp^2}{\omega^2 - k^2 v_H^2} \vec{B}_1 - \frac{p_1}{\mu_0} \vec{k} \times \vec{g},$$  \hspace{1cm} (14)

where $k_\perp = (k_x + i k_y)^{1/2}$ is the perpendicular wave number. Similarly, from Eq. (12), one finds

$$\vec{v}_1 = -\vec{k} \times \vec{B}_0 - \frac{\vec{k} \times \vec{g}}{\mu_0}.$$  \hspace{1cm} (15)

The perturbed heat flux is

$$\vec{Q}_1 = -\chi_L \left[ \vec{B}_1 \cdot (\vec{b} \cdot \nabla T_0) + \vec{b} \cdot (\vec{B}_1 \cdot \nabla T_0) + \vec{b} \cdot (i \vec{k} T_0) \right] - i k \chi_R T_1,$$  \hspace{1cm} (16)

in which $\vec{b}_0 = \vec{B}_1 / B_0 - \vec{B}_0$. It should be specifically pointed out that $\delta B$ here is defined as $|\vec{B}_1 + \vec{B}_0| - B_0$, the perturbation to the magnitude of the magnetic field, rather than $|\vec{B}_1|$, the magnitude of perturbed magnetic field. As a result,

$$\delta B = \frac{\vec{B}_0 \cdot \vec{B}_1}{B_0}.$$  \hspace{1cm} (17)

We consider $b_\perp = 0$, and hence $\vec{B} \cdot \nabla T_0 = 0$. In view of that $T_1 = -p_1 T_0 / \rho_0$, the perturbed energy equation is

$$(i \omega - \omega_T) \frac{\rho}{\rho_0} = -\frac{N^2}{g} \frac{v_{1z}}{2} + \frac{2 \chi T_0}{5 p_0 B_0} (i \vec{k} \cdot \vec{b}) (\vec{B}_1 \cdot \nabla \ln T_0),$$  \hspace{1cm} (18)

where we have denoted

$$\zeta = 1 + \frac{\chi_R}{\chi_C \cos^2 \alpha},$$  \hspace{1cm} (19)

$$N^2 = \frac{3}{5g} \frac{d}{dz} \ln \rho_0 \rho_0^{-1},$$  \hspace{1cm} (20)

and

$$\omega_T = \frac{2 \chi C_T k^2}{5 p_0} \cos^2 \alpha.$$  \hspace{1cm} (21)

Now we define a local dynamic frequency $\omega_d = \sqrt{g/L}$. For $\omega_T \gg \omega_d$, the DR reduces to

$$(\omega_\perp \omega_\perp^2 - \omega_\perp \omega_\perp^2) (\omega_\perp^2 - \omega_\perp^2 - \omega_\perp^2 - \omega_\perp^2) = 0,$$  \hspace{1cm} (22)

in which we define $\omega_\perp^2 = g \ln T_0 / dz$, which can be either positive or negative, and $\theta = k^2 / L$. When $\omega_T \gg 1$, the anisotropic effects are negligible and the Hall term can be neglected compared to the Ohmic term. Note that $\chi_C = 0$ and $\zeta \rightarrow \infty$, so that there is no MTI. That is, the anisotropic heat conducting and Hall effect become significant only when $\omega_T \gg 1$. Besides, it is known that the generalized Ohm’s law Eq. (7) is not rigorous. It treats perpendicular (with respect to the magnetic field) electron dynamics accurately but misses some effects of parallel electron dynamics, such as parallel temperature gradients and electron stress anisotropic. The Ohmic term is not right and the physical resistivity is anisotropic, with $\eta_\perp = 1.98 \eta_\parallel$. However, some space and high temperature laboratory plasmas are almost collisionless. Other terms except the resistivity term in the generalized Ohm’s law are important. Hence, the analysis above may be not right but can provide a short-cut to physical insight. In this case, from the dispersion Eq. (22), one has

$$\omega^2 = \omega_\perp^2 \zeta^{-1} \theta + k^2 v_H^2.$$  \hspace{1cm} (23)

The Hall term shows a stabilizing effect on the MTI. On the other hand, if the plasma is collisional, $\omega_T \ll 1$, but mean-
while $|\omega_{\text{min}}| \gg \eta_k^2$ so that $\zeta^{-1} \omega_{\text{min}}^2 \sim O(\eta_k^2)$ still holds, the DR [Eq. (22)] gives

$$\omega \bar{\omega} = \omega_{\Lambda}^2,$$

and

$$\omega \bar{\omega} = \omega_{\Lambda}^2 + \zeta^{-1} \omega_{\text{min}}^2 \bar{\theta}.$$  \hfill (25)

It is easy to see that the former mode is the shear Alfven wave with resistivity effect, which is always stable. The latter one is the MTI modified by resistive effect. The instability occurs if $\omega_{\Lambda}^2 + \zeta^{-1} \omega_{\text{min}}^2 \bar{\theta} < 0$. The resistivity does not alter the criterion but decreases the growth rate.

### IV. Anisotropic Resistivity Modes

When the magnetic field is sufficiently strong (compared to the collision), the heat conduction, resistivity, and viscosity are all anisotropic. We suppose that the kinetic pressure $p$ is isotropic. For $\omega_c r \gg 1$, the Ohm’s law should read\textsuperscript{17}

$$\vec{J} = \sigma \cdot (\vec{E} + \vec{v} \times \vec{B}),$$ \hfill (26)

where

$$\sigma = \eta_{\perp}^{-1} (I - \hat{b} \hat{b}) + \eta_{\parallel}^{-1} \hat{b} \hat{b},$$ \hfill (27)

where $I$ is the unit tensor, $\eta_{\perp}$ is the CR, and $\eta_{\parallel}$ is the PR. Accordingly, we can rewrite the Ohm’s law as the following form,

$$\vec{E} + \vec{v} \times \vec{B} = \eta_{\perp} \vec{J} + \eta_{\parallel} \vec{J} = \eta_{\perp} \vec{J} - \Delta_{\phi} \hat{b} (\hat{b} \cdot \vec{J}),$$ \hfill (28)

where $\Delta_{\phi} = \eta_{\perp} - \eta_{\parallel} = \eta_{\parallel}$. By virtue of Eq. (3), we obtain the perturbed magnetic field

$$\vec{B}_1 = -\frac{\vec{k} \cdot \vec{B}_0}{\omega_0} \vec{v}_1 - \frac{\Delta_{\phi}(\vec{k} \cdot \vec{B}_0)(\vec{\phi} \cdot \vec{v}_1)}{\omega_0 (\omega - i \Delta_{\phi} \omega^2)},$$ \hfill (29)

in which $\vec{\phi} = \vec{k} \times \vec{b}$ and $\omega_0$ is now defined as $\omega + i \eta_{\parallel} \vec{k}^2$.

For viscosity, it is appropriate to adopt the Braginskii’s expression for the viscosity tensor $\bar{\Pi}$\textsuperscript{1,18-20}

$$\bar{\Pi} = \nu_0 (3b \vec{b} - I)[\hat{b} \cdot (\hat{b} \cdot \nabla) \vec{v} - \frac{1}{3} \nabla \cdot \vec{v}],$$ \hfill (30)

where $\nu_0$ is the first Braginskii’s coefficient of viscosity. In the Boussinesq approximation, one gets $\partial_v \vec{v}_1 = 0$ and hence, the viscosity term in Eq. (4) is simplified to

$$\Pi_{ij} \partial_v v_{ij} = \nu_0 (3b \vec{b} - I)[\hat{b} \cdot (\hat{b} \cdot \nabla) \vec{v} - \frac{1}{3} \nabla \cdot \vec{v}].$$ \hfill (31)

The perturbed motion equation is now

$$-i \omega_0 \vec{v}_1 = -ik\left[ p + \frac{\vec{B}_0 \cdot \vec{B}_0}{\mu_0} \right] + \frac{i}{\mu_0} (\vec{k} \cdot \vec{B}_0) \vec{B}_1 + \rho_1 \vec{g} + \nu_0 (\vec{b} \cdot \vec{v}_1)(\vec{k} \cdot \vec{b})[\vec{k} \times b(\hat{k} \cdot \vec{b})].$$ \hfill (32)

The wave vector $\vec{k}$ timing this formula yields

$$-i \omega \vec{k} \times \vec{v}_1 = \frac{i}{\rho_0 \mu_0} (\vec{k} \cdot \vec{B}_0) \vec{k} \times \vec{B}_1 + \rho_1 \vec{g}$$

$$- \omega_0 \bar{\omega} (\vec{b} \cdot \vec{v}_1),$$ \hfill (33)

in which $\omega_0 = 3\nu_0 k^2 \alpha / \rho_0$ is the viscosity frequency. Inserting $\vec{B}_1$, one has

$$-i(\omega - \omega_{\Lambda}^2 / \omega) \vec{k} \times \vec{v}_1 = \rho_1 \vec{k} \times \vec{g} + \frac{\omega_{\Lambda}^2 \Delta_{\phi}(\vec{\phi} \cdot \vec{v}_1) \vec{k} \times \vec{\phi}}{\omega_0 (\omega - i \Delta_{\phi} \omega^2)}$$

$$- \omega_0 \bar{\omega} (\vec{b} \cdot \vec{v}_1).$$ \hfill (34)

Now, $\vec{b} [\text{Eq. (34)}]$ and $\vec{k} [\text{Eq. (34)}]$ yield

$$\vec{\phi} \cdot \vec{v}_1 = \frac{\rho_1}{\rho_0} (\vec{\phi} \cdot \vec{g}) \frac{\omega_0 - i \Delta_{\phi} \omega^2}{\omega_0 (\omega - i \Delta_{\phi} \omega^2)} - \omega_{\Lambda}^2 \vec{k} \times \bar{\omega} (\vec{b} \cdot \vec{v}_1),$$ \hfill (35)

and

$$-ik^2 \left[ (\omega - \omega_{\Lambda}^2 / \omega) \vec{v}_1 = \frac{\rho_1}{\rho_0} \vec{k} \times (\vec{k} \times \vec{g}) - \frac{\Delta_{\phi} k^2 \omega_{\Lambda}^2 (\vec{\phi} \cdot \vec{v}_1) \vec{\phi}}{\omega_0 (\omega - i \Delta_{\phi} \omega^2)} \right]$$

$$- \omega_0 \bar{\omega} (\vec{b} \cdot \vec{v}_1),$$ \hfill (36)

respectively. Substituting $\vec{\phi} \cdot \vec{v}_1$, we obtain from Eq. (36) that

$$ik^2 \left[ (\omega - \omega_{\Lambda}^2 / \omega) \vec{v}_1 = \frac{\rho_1}{\rho_0} \vec{k} \times (\vec{k} \times \vec{g}) - \frac{i \Delta_{\phi} k^2 \omega_{\Lambda}^2 (\vec{\phi} \cdot \vec{v}_1) \vec{\phi}}{\omega_0 (\omega - i \Delta_{\phi} \omega^2)} \right]$$

$$- \omega_0 \bar{\omega} (\vec{b} \cdot \vec{v}_1),$$ \hfill (37)

Again, adopting $\vec{b} [\text{Eq. (37)}]$, one has

$$\vec{b} \cdot \vec{v}_1 = \frac{\rho_1}{\rho_0} \frac{\vec{\phi} \cdot (\vec{k} \times \vec{g})}{ik^2 \omega^2 - \omega_0 \bar{\omega}^2}.$$ \hfill (38)

Substituting this formula into Eq. (37), we arrive at the perturbed velocity,

$$\vec{v}_1 = \frac{\rho_1 \bar{\omega}}{i \rho_0 k^2 \omega^2} \left[ \vec{k} \times (\vec{k} \times \vec{g}) - \frac{i \omega_0 \Delta_{\phi} k^2 (\vec{\phi} \cdot \vec{g}) \vec{\phi}}{\omega_0 (\omega^2 - i \omega \Delta_{\phi} \omega^2)} \right] + \frac{\omega_0 \bar{\omega} k \times \vec{\phi}(\vec{k} \times \vec{g} \cdot \vec{\phi})}{ik^2 \omega^2 - \omega_0 \bar{\omega}^2},$$ \hfill (39)

and then,

$$\vec{B}_1 = \frac{\bar{\omega}}{\omega^2} \rho_1 \left[ \frac{\vec{k} \cdot (\vec{k} \times \vec{g})}{k^2} - \frac{i \omega_0 \Delta_{\phi} (\vec{\phi} \cdot \vec{g}) \vec{\phi}}{\omega_0 (\omega^2 - i \omega \Delta_{\phi} \omega^2)} \right]$$

$$+ \frac{\omega_0 \bar{\omega} k \times \vec{\phi}(\vec{k} \times \vec{g} \cdot \vec{\phi})}{k^2 (ik^2 \omega^2 - \omega_0 \bar{\omega}^2)},$$ \hfill (40)

where $\omega^2$ stands for $\omega^2 - \omega_0 \bar{\omega}^2$.

Finally, the energy equation is
where we used $\xi = 1$ by neglecting the isotropic heat conduction. Recall that $\tilde{g} = -g\tilde{z}$, from the expression of $\tilde{B}_1$, we have

$$\tilde{B}_1 = i\tilde{k} \cdot \tilde{B}_0 \frac{\rho_1}{\rho_0} \frac{\tilde{g} \times \tilde{g}}{\omega^2 + i\omega_\nu \omega_\nu^2 / k^2},$$

as well as

$$v_{1c} = \frac{\rho_1 \tilde{\omega}}{i\rho_0 k \omega} \left[ k_x^2 g + \frac{i\omega_\nu^2 \Delta_\nu g}{\omega} (\omega^2 - i\omega_\nu \nu^2) \right] + \frac{\omega_\nu \tilde{\omega}(k_x \varphi_y - k_y \varphi_x)^2}{k^2 (i k^2 \omega^2 - \omega_\nu \omega_\nu^2)} \frac{1}{\omega^2 + i\omega_\nu \omega_\nu^2 / k^2},$$

according to Eq. (39). Substituting these three formulas above into Eq. (41) yields the DR

$$(\omega_T - i\omega) \omega^2 = -i\omega N^2 \left[ k_x^2 k_x^2 + \frac{i\omega_\nu^2 \Delta_\nu g}{\omega} (\omega^2 - i\omega_\nu \nu^2) \right] + \frac{\omega_\nu \tilde{\omega}(k_x \varphi_y - k_y \varphi_x)^2}{k^2 (i k^2 \omega^2 - \omega_\nu \omega_\nu^2)} \frac{1}{\omega^2 + i\omega_\nu \omega_\nu^2 / k^2}$$

$$\times \left[ k_y^2 k_y^2 + \frac{i\omega_\nu^2 \Delta_\nu g}{\omega} (\omega^2 - i\omega_\nu \nu^2) \right] + \omega_T \omega^2_{\text{mit}}$$

$$+ \frac{\omega_\nu \tilde{\omega}(k_x \varphi_y - k_y \varphi_x)^2}{k^2 (i k^2 \omega^2 - \omega_\nu \omega_\nu^2)} \frac{1}{\omega^2 + i\omega_\nu \omega_\nu^2 / k^2} \frac{(k_x \varphi_y - k_y \varphi_x)^2}{k^2}$$

$$+ \frac{2 b_z (k_x \varphi_y - k_y \varphi_x) \omega^2}{k^2 (\omega^2 + i\omega_\nu \omega_\nu^2 / k^2)}.$$  

This DR describes the thermal convective instability with background heat flux in the presence of anisotropic resistivity and viscosity in a thermally laminar plasma. In the ideal MHD limit, viz., $\eta = \nu = 0$ and $v_0 = 0$, one finds $\tilde{\omega} = \omega$, $\Delta_\eta = 0$, and $\omega_\nu = 0$. The DR is simplified to

$$(\omega + i\eta k_x^2) (\omega^2 - i\omega_\nu \nu^2) \omega^2_{\text{mit}} - \omega^2 = 0.$$  

$$(\omega + i\eta k_x^2) (\omega^2 - i\omega_\nu \nu^2) \omega^2_{\text{mit}} - \omega^2_{\text{mit}} = 0.$$  

The growth rate $\gamma$ is

$$\gamma = \frac{1}{\eta} \left[ (\omega + i\eta k_x^2) (\omega^2 - i\omega_\nu \nu^2) \omega^2_{\text{mit}} - \omega^2_{\text{mit}} \right].$$  

According to the growth rate, the instability takes place when

$$\omega^2_{\text{mit}} < -\eta \nu^2 \omega_\nu - \omega^2_\lambda.$$  

One can see that in this case, only the CR and the viscosity affect the instability growth rate and criterion. The PR does not contribute to them. In the criterion, the CR coupled with the viscosity plays a damping effect on the HBI. Neither of them shows effects on the instability criterion when either of them is equal to zero and consequently, the instability criterion is reduced to $\omega^2_{\text{mit}} < -\omega^2_\lambda$. However, the instability growth rate Eq. (52) is always diminished by either the CR or the viscosity. This is consistent with the previous results.10,11 Introducing the magnetic Prandtl number $P_m$, a material property of the fluid, is defined as $\omega_\nu \nu^2 / (\eta k_x^2)$.
here. In the special case of \( P_m=1 \), the growth rate becomes
\[
\gamma = -\omega_{\text{mii}}^2\xi - \omega_{\text{A}}^2 - \omega_{\text{b}}^2/2k^2.
\]

(54)

B. \( \nu, y=0, b_y=0 \)

For \( \nu, y=0 \) and \( b_y=0 \), that is, \( k_x/k_z=b_x/b_z \), meaning that the projection of wave vector \( \vec{k} \) on the \( x-z \) plane is parallel to the magnetic field, we have
\[
(\omega^2 - \omega_{\text{mii}}\xi - \omega_{\text{A}}^2) - \omega_{\text{b}}^2/2k^2 = 0.
\]

(55)

This is a quartic equation about the frequency \( \omega \). It can be numerically solved to examine the effects of anisotropic resistivity and viscosity on the buoyancy instability and needs further simplification for analytic analysis. Supposing the Alfvén frequency is much less than \( \omega_{\text{A}} \), one finds \( \omega^2 = \omega_{\text{A}} \) and the DR above goes to
\[
\omega_0(\omega + i\omega_{\text{mii}}k^2/2)(\omega - i\Delta \xi) - \omega_{\text{A}}\omega_{\text{mii}}^2 = 0.
\]

(56)

1. Resistivity effect

The resistivity effect is first examined in this part by letting \( \omega_{\text{A}}=0 \).

a. Parallel-resistivity-dominant mode Consider PR is much greater than CR, namely, \( \eta \gg \eta_1 \). Accordingly, one has \( \omega = \omega_{\text{A}} \) and \( \Delta \xi = -\eta_1 \). The dispersion Eq. (56) goes to
\[
\omega^2(\eta_1k_x^2 - i\omega_{\text{mii}}k^2) + \eta_1k_z^2(b_z^2 - \xi)\omega_{\text{mii}}^2 = 0.
\]

Now we introduce a frequency \( \omega = \eta_1k_x^2 \) and then define three dimensionless parameters: normalized growth rate, \( \eta_1 = -\omega_{\text{mii}}/\omega_{\text{A}} \), normalized resistivity frequency, \( S = \omega_{\text{mii}}/\omega_{\text{A}} \), and \( \xi = \omega_{\text{mii}}^2/\omega_{\text{A}}^2 \). The DR above is normalized to
\[
\eta_1^2 + \eta_1^2S + \eta_1^2\xi^2 - (b_z^2 - \xi)S\xi^2 = 0.
\]

(58)

When \( \omega_{\text{A}}=0 \), viz. in the ideal MHD limit, the two terms \( \eta_1^2 \) and \( \eta_1^2\xi^2 \) are kept down while the others become zero, leading to the instability condition \( \xi^2 < 0 \) with a growth rate \( \eta_{\text{ideal}} = (-\xi^2)^{1/2} \). The left-hand side (lhs) of the equation above is a function that increases monotonically with \( \gamma \) when \( \xi^2 > 0 \). Hence, there exists a positive root for \( \gamma \) provided that this function is less than zero at \( \gamma=0 \), and so the instability emerges if \( (b_z^2 - \xi)^2 < 0 \). Before continuing, we need to judge the range of the value of \( \xi \). With the help of the expression of \( \xi \), letting \( b_z = \cos \theta \) and \( b_z = \sin \theta \), where \( \theta \) is the angle between the magnetic field \( \vec{B}_0 \) and the vertical direction, we find \( k_z = \sqrt{k_x^2 + k_y^2}\sin \theta \), \( k_y = \sqrt{k_x^2 + k_y^2}\cos \theta \), and then we have
\[
b_z^2 - \xi = \sin \theta^2 (1 - 2\cos \theta)\sqrt{k_x^2 + k_y^2}\sin \theta^2 + \frac{k_y^2}{k_x^2 + k_y^2}
- 2\sin^2 \theta \cos^2 \theta - \frac{k_y^2}{k_x^2 + k_y^2} = \cos^2 \theta - \frac{k_z^2}{k_x^2 + k_y^2},
\]

(59)

and

\[
\xi = \cos^2 \theta \left( \tan^2 \theta - \frac{k_z^2}{k_x^2 + k_y^2} \right).
\]

(60)

Thus, the maximums of \( b_z^2 - \xi \) and \( \xi \) are \( k_x^2/k_z^2 \) and \( b_z^2 \), respectively, and \( b_z^2 - \xi \) is always positive. In this case, the instability criterion is simplified to \( \xi^2 > 0 \). Meanwhile, note that \( \xi^2 > 0 \) requires \( \xi^2 > 0 \).

On the other hand, when \( \xi^2 < 0 \), we cannot directly derive the instability criterion by using the same judging method. We consider some limiting cases. For \( S \ll |\xi| \), the dispersion Eq. (58) has a root for \( \gamma \) on the order of \( S \) and a root on the order of \( \xi \). For the former (small) one, the last two terms on the lhs of Eq. (58) dominate, giving rise to the growth rate \( \gamma_{p1} = (b_z^2 - \xi)/S \) and the instability criterion is \( (b_z^2 - \xi)^2 < 0 \), namely, \( \xi^2 < 0 \). This is an unstable mode purely induced by the resistivity effect, which disappears in the ideal MHD limit. Combining the two cases together, we find that an unstable mode comes into being in the presence of resistivity effect, and the instability criteria are

\[
\xi^2 > 0, \quad \text{when } \xi > 0;
\]

\[
\xi^2 < 0, \quad \text{when } \xi > 0 \text{ (Note that } \gamma \text{ is small).}
\]

(61)

Therefore, the critical condition for instability to occur is \( \xi^2 > 0 \) and it is independent of the sign of \( \xi^2 \). In fact, the same conclusion can be obtained directly from the expression of the growth rate \( \gamma_{p1} \) (note that \( b_z^2 - \xi \) and \( S \) are both positive). The presence of resistivity not only affects the growth rate but also changes the instability criteria. Since this unstable mode is purely induced by resistivity, it does not occur in the ideal MHD case (for \( S = 0 \), \( \gamma_{p1} = 0 \)). For the root with the order of \( \xi \), according to Eq. (58) we have \( \gamma_{p2} = (-\xi^2)^{1/2} - b_z^2/(2S) \) \( \eta_{\text{ideal}} - b_z^2 S/(2\xi) \), which is now positive. This mode is also unstable. The resistivity is shown to reduce the growth rate when \( \xi > 0 \) and increase the growth rate when \( \xi < 0 \).

Another limiting case is \( S \gg |\xi| \), in which there are two roots for \( \gamma \). One is \( \gamma_{p1} = -S + b_z^2 S/(2\xi) \) with the order of \( S \) and the other is \( \gamma_{p2} = \sqrt{(b_z^2 - \xi)^2 - b_z^2 S}(2S) \), with the order of \( \xi \). It is not hard to see that the first root represents a damping mode since \( S \) is always positive, and the instability criterion for the second mode is \( \xi^2 > 0 \), differing from that in the ideal MHD case. In summary, the critical conditions for instability are listed in Table I.

Now we numerically examine the effect of resistivity on the buoyancy instability with the help of dispersion Eq. (58). Letting \( b_z^2 = 1/2 \) (\( \theta = \pi/4 \)), for \( \xi = 0.2 \) and \( \xi = 4 \), we find \( \gamma = 0.26 \) for \( S = 0 \) and \( \gamma = 0.91 \) for \( S = 4 \) for \( \xi = 0.2 \) and

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(62)
$\xi^2 = -4$, one finds $\gamma = 0.16$ for $S = 0.1$ and $\gamma = -4.43$ for $S = 4$. Letting $b_0^2 = 1/4 (\theta = \pi/6)$, for $\xi = -0.2$ and $\xi^2 = -4$, there is $\gamma = -0.41$ for $S = 0.2$ and $\gamma = -4.22$ for $S = 4$; for $\xi = -0.2$ and $\xi^2 = 4$, we obtain $\gamma = 0.98$ for $S = 0.2$ and $\gamma = 1.25$ for $S = 4$. These results confirm our analysis. However, it should be specifically stressed that only the purely unstable mode is considered in our analysis. The modes with complex frequency $\omega = \omega_0 + i \gamma$, where $\omega_0$ is a real number, may lead to a final conclusion but is beyond the scope of the present work.

Finally, in the limit of $\xi = 0$, viz. $k_y = k \tan (\theta)$ ($\theta$ should be less than $\pi/4$) to make sure that $k_y < k$ is satisfied, $\gamma = \pi/2$ is an extraneous root (which is dropped), for which there is no instability in the ideal MHD case since the growth rate $\gamma_{\text{ideal}}$ is proportional to $|\xi|^{1/2}$. In the presence of resistivity, the dispersion Eq. (58) is reduced to

$$\psi^3 + \psi^2 S - S \xi^2 \sin^2 \theta = 0. \quad (62)$$

For $S \ll |\xi|$, $\gamma_{\text{es}} = (S \xi^2 \sin^2 \theta)^{1/3} - S/3$ and for $S \gg |\xi|$, the growth rate is determined by $\gamma_{\text{es}}$. Hence, the instability criterion is $\xi^2 > 0$ in the two limiting cases. In fact, since the lhs of the equation above increases monotonically with $\gamma$, we arrive at the instability criterion immediately, $-S \xi^2 \sin^2 \theta < 0$, namely, $\xi^2 > 0$. It is known that when $\xi^2 > 0$, i.e., when $dT/d\xi > 0$, the system is MTI stable but HBI unstable. Thus, to some extent, we can say that the unstable mode in this case induced by the resistivity is a HBI mode rather than a MTI mode.

b. Spitzer Mode According to the Spitzer resistivity, $\Delta_\eta = \eta_\perp/2$. Defining $\gamma_{\eta} = \eta_\perp k^2$ and $\gamma_{\Delta} = \Delta_\eta k^2 = k^2/(2k^2) \gamma_{\eta}$, the dispersion Eq. (56) can be expressed as

$$\psi^3 + \psi^2 (2 \gamma_{\eta} - \gamma_{\Delta}) + \gamma_\Delta \gamma_{\eta} \gamma_{\Delta} + \gamma_{\Delta} b_0^2 = 0. \quad (63)$$

In the case of $\omega_{\text{es}}^2 > \gamma_{\eta} \gamma_{\Delta} - \gamma_{\eta}$, which can be easily satisfied when $\gamma_{\eta}$ is large enough and/or $\omega_{\text{es}}^2$ is positive, the coefficient of the first-order term is positive and hence the lhs of the equation above is a monotonic function that increases as $\gamma_{\eta}$ increases. The critical condition corresponding to instability is

$$\omega_{\text{es}}^2 (\gamma_{\eta} \xi - \gamma_{\Delta} \xi + \gamma_{\Delta} b_0^2 < 0. \quad (64)$$

Recall that $\xi = \sin^2 \theta - \cos^2 \theta k^2 / k_x^2$; one has

$$\gamma_{\eta} \xi - \gamma_{\Delta} \xi + \gamma_{\Delta} b_0^2 = \sin^2 \theta - \cos^2 \theta k^2 / k_x^2 \left( \frac{k^2}{k_x^2} - \frac{k^2}{2k_x^2} \right), \quad (65)$$

which is negative when $\theta \rightarrow 0$ and positive when $\theta \rightarrow \pi/2$. Thus, $\omega_{\text{es}}^2$ can be either positive or negative for instability. For comparison with the ideal criterion $\omega_{\text{es}}^2 \xi < 0$, we re-express the condition as

$$\omega_{\text{es}}^2 \xi < -\frac{\gamma_{\Delta} b_0^2}{\gamma_{\eta} - \gamma_{\Delta}} \omega_{\text{es}}^2. \quad (66)$$

The resistivity shows a significant effect on the HBI. The instability criterion is remarkably altered by the presence of parallel and cross-field resistivity.

For $\omega_{\text{es}}^2 < \gamma_{\eta} \gamma_{\Delta} - \gamma_{\eta}$, it is hard to obtain the criterion directly. Similarly, when we assume $\gamma_{\eta} \ll |\omega_{\text{es}}|$, the dispersion Eq. (63) has a root for $\gamma$ on the order of $\omega_{\text{es}}$ (the ideal one) and a root on the order of $\gamma_{\eta}$. For the smaller one, the last two terms on the lhs of Eq. (63) dominate, giving rise to the growth rate $\gamma = -\gamma_{\Delta} (b_0^2 - \xi^2) / \xi^2 - \gamma_{\eta}$. The growth rate indicates that this mode is a damping mode and cannot develop into instability, while this mode is unstable when $\gamma_{\eta} = 0$ and $\gamma_{\Delta} < 0$ as we have discussed above (see the expression of $\gamma_{\eta}$).

Resistive effects are believed to be very interesting in the presence of internal boundary layers (current sheets) and/or associated with the magnetic reconnection process in space and laboratory plasmas and have been shown to be important in the context of magnetorotational instability (MRI). In many astrophysical situations, especially in dilute plasmas, the assumption of ideal MHD, i.e., infinite conductivity so that the magnetic field is frozen-in to the plasma of zero resistivity, does not hold. The plasma in such situations should be characterized by finite resistivity. However, in weakly collisional plasmas, generally the anisotropic heat transport is the fastest phenomenon in the evolution of the system, which is the dominant mode while other dissipation effects are much weaker. The parallel heat transport introduces a time scale much shorter than the other typical time scales characterizing the system evolution. That is, $\omega_{\eta}$ is much greater than other typical frequencies, such as $\omega_{\perp}$, $\omega_{\text{es}}$, and $\omega_{\eta}$. Since we have restricted ourselves to the mode with a slow time scale characterized by $\omega_{\eta}$ by assuming $\omega_{\eta} \gg \omega_{\eta}$, the resistivity effect cannot be neglected even if it is much less than the heat transport effect, because $\omega_{\eta}$ can be of the same order as $\omega_{\eta}$. For this reason, it is necessary to examine the effects induced by the anisotropic resistivity on the buoyancy instability.

2. Viscosity effect

It is known that some astrophysical environments, such as the solar corona, are well-known examples of a plasma that should be described by an anisotropic viscous MHD model. Islam and Balbus studied the magnetoviscous instability in the dilute astrophysical plasmas environment in 2005 and then investigated this instability associated with MTI. In this part, we focus on the viscosity by letting $\omega = \omega_{\eta}$ and $\Delta_\eta = 0$. The dispersion Eq. (56) is simplified to

$$\omega (i \omega - \omega_{\perp} k^2) - \omega_{\text{es}}^2 (\xi \omega - \omega_{\perp} b_0^2 k^2 / k_x^2) = 0. \quad (67)$$

Introducing $\sigma = \omega_{\perp} k^2 / (k^2 \omega_{\eta})$ and using $\gamma = -i \omega / \omega_{\eta}$ and $\xi^2 = \omega_{\text{es}}^2 / \omega_{\eta}$, the equation above is of the form

$$\psi^3 + \psi^2 \sigma + \psi \xi^2 + b_0^2 \sigma_0^2 \xi^2 = 0. \quad (68)$$

Due to the similarity of expressions between Eqs. (58) and (68), we present a brief discussion on the equation above. The lhs of the equation above is a function that increases monotonically with $\sigma$ for positive $\xi^2$; thus, the instability criterion is $b_0^2 \sigma_0^2 \xi^2 < 0$, viz. $\xi^2 < 0$. Meanwhile, $\xi^2 > 0$ requires $\xi < 0$. Unstable modes coming into being in this case are introduced purely by the viscosity. For negative $\xi^2$, which is corresponding for instability in the ideal MHD case, we also need to simplify the dispersion relation with some assumptions. When $\sigma \ll |\xi|$, one has $\gamma_{\eta} = -b_0^2 \sigma / \xi$, which is
unstable for positive $\xi$; $\gamma_{m2} = \gamma_{\text{ideal}} + (b_x^2 - \xi^2)/(2\xi)$, implying that the viscosity increases the growth rate for positive $\xi$ and reduces the growth rate for negative $\xi$. When $\sigma \gg |\xi|$, one has $\gamma_{m1} = -\sigma - (b_x^2 - \xi^2)/\sigma$, which is a damping mode, and $\gamma_{m2} = -b_x^2\xi^2$, which is unstable when $\xi^2 < 0$. The viscosity shows a significant effect on the growth rates and instability criteria. The stability boundary is listed in Table II. In the limit of $\xi = 0$, the perturbation becomes unstable provided that $\xi^2 < 0$; thus, the unstable mode in this case induced by the viscosity is a MTI mode.

The analysis above is restricted to some limiting cases. The general critical conditions for the instability to develop are not obtained and the overstable modes are not considered. Here, we are expected to derive the expressions of the instability criteria by ignoring the growth rate. Letting $\omega = \omega_0 + iy$ in the dispersion Eq. (67), we obtain the real part

$$\gamma^3 - 3\gamma \omega_0^2 + (\gamma^2 - \omega_0^2)\sigma + \gamma \xi^2 + b_x^2\sigma \xi^2 = 0,$$

and the imaginary part,

$$\omega_0(b_x^2 - 3\gamma^2 - 2\gamma \sigma - \xi^2) = 0,$$

where $\omega_0$ is defined as $\omega_0/\omega_d$. For the pure unstable/damping modes, $\omega_0 = 0$, Eq. (69) is reduced to the dispersion Eq. (68). To obtain the instability criteria, we define the lhs of Eq. (68) as a function of $\gamma$, $F(\gamma)$. As a cubic function of $\gamma$, there may be two flex points determined by $\partial F/\partial \gamma = 0$, which reads

$$\frac{\partial F}{\partial \gamma} = 3\gamma^2 + 2\gamma \sigma + \xi^2.$$

Letting the equation above equal zero, we have the two flex points at

$$\gamma_{c1} = \frac{-\sigma + \sqrt{\sigma^2 - 3\xi^2}}{3},$$

and

$$\gamma_{c2} = \frac{-\sigma - \sqrt{\sigma^2 - 3\xi^2}}{3},$$

requiring $\xi^2 < \sigma/3$. When $\xi^2 \geq \sigma/3$, $F(\gamma)$ is a monotonic function increasing with $\gamma$. After some algebraic analyses, we find the unstable domain corresponding to the following conditions,

$$\begin{cases}
F(0) < \gamma_{c1}(2\gamma_{c1} + \sigma), & \text{for } \xi^2 < 0 \\
F(0) < 0, & \text{for } \xi^2 > 0
\end{cases}$$

in which $F(0) = b_x^2\sigma \xi^2$.

As for the overstable modes (both $\gamma$ and $\omega_d$ are nonzero), Eq. (70) gives

$$\omega_0^2 = 3\gamma^2 + 2\gamma \sigma + \xi^2.$$

Substituting it into the real part leads to

$$D(\gamma) = 8\gamma^3 + 8\gamma^2 \sigma + 2\gamma(\xi^2 + \sigma^2) + D(0) = 0,$$

with $D(0) = -\sigma \xi^2 \cos^2 \theta k_y^2 l^2$. After a similar analysis, we find the instability criteria in the form

$$\begin{cases}
D(0) + 2\gamma_{c1}(\gamma_{c1} + \sigma) < 0, & \text{for } \xi^2 < 0 \\
D(0) < 0, & \text{for } \xi^2 > 0
\end{cases}$$

Recall that $\xi^2 > 0$ corresponds to stability in the ideal MHD case; the presence of viscosity is shown to not only modify the ideal instability criteria but also introduce new unstable modes. We also note that the overinstabilities are purely introduced by the viscosity and do not exist in the ideal MHD case since if $\sigma = 0$, $\omega_0^2 = \xi^2/4$ is negative so that there is no self-consistent wave mode.

### 3. Viscoresistive case

Now we consider resistivity and viscosity simultaneously. We still assume $\eta_\text{c} \ll \eta$ and have

$$(\gamma + \sigma)(\gamma + S)\gamma + \gamma \xi^2 + [S(\xi - b_x^2) + \sigma b_x^2] \xi^2 = 0.$$ (78)

Since the classical criterion for instability is $\xi^2 < 0$, we focus on the case of $\xi^2 > 0$ to examine the dissipative effect on the instability criterion. Similarly, the lhs of the equation above increases monotonically with $\gamma$ now, yielding the instability criterion,

$$[S(\xi - b_x^2) + \sigma b_x^2] \xi^2 < 0.$$ (79)

When $\xi^2 < 0$, $\xi$ is limited to be negative and the inequality above gives $\xi > (1 - \sigma/S)b_x^2$ so, $\sigma/S > 1$ should be satisfied. When $\xi^2 > 0$, the inequality gives $\xi < (1 - \sigma/S)b_x^2$ combined with $\xi > 0$, yielding $\sigma/S < 1$. That is, $\sigma/S < 1$ or $\sigma/S > 1$, a new unstable mode is excited. In other words, for $\sigma = S$, the dissipation effect cannot alter the instability criterion and induce new unstable modes. In fact, the same conclusion can be directly obtained from the dispersion Eq. (78). We define $P_m = \sigma/S$ as the magnetic Prandtl number in this part. Letting $P_m = 1$, from Eq. (78) we have

$$\gamma^2 + \gamma \sigma + \xi^2 = 0,$$

leading to the growth rate

$$\gamma = \sqrt{-\xi^2 + \frac{\sigma^2}{4} - \frac{\sigma}{2}},$$

which implies that the dissipation terms have no effect on the instability criterion in this case. The instability criterion is still $\xi^2 < 0$ and the anisotropic dissipative effect just reduces the value of the growth rate. No new unstable modes are introduced. The same conclusion can be generated from Eq. (54), that when the magnetic Prandtl number $P_m = 1$, dissipation effects do not play a role in the instability criteria.

Finally, in the presence of PR, CR, and viscosity simultaneously, we give a rough and short discussion by letting

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**TABLE II. Critical condition for instability in the presence of viscosity.**

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b_r=0, namely, considering only the vertical magnetic field. The DR Eq. (56) is reduced to a cubic equation,
\[ y^3 + \gamma_r^2(2\gamma_r + \gamma_r - \gamma_\Delta) + \gamma_r(\gamma_r - \gamma_\Delta)(\gamma_r + \gamma_r) + \gamma_r\gamma_r + \omega_{mi}^2 \xi) = 0. \] 
(82)
Assuming the coefficient of the first-order term is positive, we obtain one of the unstable regions as
\[ -(\gamma_r - \gamma_\Delta)(\gamma_r + \gamma_r) - \gamma_r\gamma_r < \omega_{mi}^2 \xi < -\gamma_r\gamma_r. \] 
(83)

VI. CONCLUSION

We investigate the linear buoyancy instability in the presence of anisotropic resistivity and viscosity. The background heat flux is taken into account in our calculation. By adopting the WKB approximation and Boussinesq limit, we derive the DR Eq. (22) by using the Ohm’s law Eq. (7) containing the isotropic resistivity and Hall term. This term becomes important only when \( \omega_{r} \gg 1 \), while in this situation the Ohmic term is actually not physically right. It just provides a short-cut to physical insight. Equation (23) shows that the Hall term exerts a damping effect on the MTI.

By assuming that the Ohmic term comprises PR and CR, we deduce the DR as shown in Eq. (47) for \( \omega_r \gg \omega_e \). Two special cases, \( b_r=0 \) and \( \rho_r=0 \) with \( b_r=0 \), are discussed in detail. For \( \rho_r=0 \) and \( b_r=0 \), the reduced DR Eq. (50) indicates that only the CR and viscosity exert effects on the DR, which contains no effect of PR. The critical condition for the occurrence of HBI is modified by the CR coupled with the viscosity, as shown in Eq. (53). The viscosity and CR can also reduce the value of the growth rate expressed in Eq. (52). For \( \rho_r=0 \) and \( b_r=0 \), the DR Eq. (47) is reduced to Eq. (56) in the weak fields limit, which is simplified and then normalized to Eq. (58) by concentrating on the PR effect while disregarding the CR and viscosity, and to Eq. (68) when the resistivity is disregarded. The PR (resp. viscosity) is shown to significantly modify the instability criterion and growth rate. It can also introduce a new unstable mode, which cannot come into being in the ideal MHD case. Considering only the viscosity, we present the analytic expressions of the instability criteria. For the purely unstable modes, the instability takes place provided that Eq. (74) is satisfied. For the overinstabilities, the imaginary part of the wave frequency \( \omega \) is positive when criteria Eq. (77) is satisfied.

When the CR and PR are both taken into account, the reduced DR Eq. (63) shows that the instability comes into being provided that Eq. (66) is satisfied when \( \omega_{mi}^2 > \gamma_r\gamma_\Delta - \gamma_r^2 \). When \( \gamma_r \) is much less than \( \omega_{mi} \), the resistivity is shown to induce a damping mode, which is unstable in the case of when only PR is in presence. In the case of \( \sigma \neq 0 \) as well as \( S \neq 0 \) when disregarding the CR, it is found from the dispersion Eq. (78) that the presence of dissipation effects will alter the instability criterion and introduce a new unstable mode when \( \sigma \neq S \). When \( \sigma = S \), the instability criterion stays the same with the one in the ideal MHD limit and only the growth rate is diminished. When CR, PR, and viscosity are all taken into account, the inequality Eq. (83) implies that a new unstable region is introduced. Our results may be of more immediate relevance for the interpretation of astrophysical simulations than actual astrophysical environments and can help understand simulations especially those performed at lower magnetic Reynolds number regimes.

ACKNOWLEDGMENTS

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