Market Models for Forward Swap Rates and Credit Default Swap Spreads

Marek Rutkowski
School of Mathematics and Statistics
University of New South Wales
Sydney, Australia

Joint work with Libo Li

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We would like to address the following issues.

1. A unified approach to modelling of market rates.
2. A comparison of the top-down and bottom-up approaches.
4. Positivity of rates within market models.
5. Construction of default times consistent with CDS spreads.
The following problems arise in the context of market models.

1. The choice of admissible families of swaps and/or credit default swaps.
2. Computation of Radon-Nikodým densities for martingale measures.
3. Computation of dynamics of forward swap rates and CDS spreads under swap/CDS measures.
4. Existence and construction of default times consistent with CDS spreads.
Families of Forward Swaps
Terminology and notation:

1. Let $0 < T_0 < T_1 < \cdots < T_{n-1} < T_n$ be a fixed sequence of dates representing the tenor structure $\mathcal{T}$.
2. We denote $a_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$.
3. Let $B(t, T_i)$ stand for the price of the zero-coupon bond maturing at $T_i$.
4. Let $\mathcal{S} = \{S_1, \ldots, S_l\}$ be any family of $l$ distinct forward swaps associated with the tenor structure $\mathcal{T}$.
5. Any reset or settlement date for any swap $S_j$ in $\mathcal{S}$ belongs to $\mathcal{T}$. 
The forward swap rate $\kappa^j_t$ for the forward swap $S_j$ starting at $T_{s_j}$ and maturing at $T_{m_j}$ equals

$$\kappa^j_t = \kappa^{s_j, m_j}_t = \frac{B(t, T_{s_j}) - B(t, T_{m_j})}{\sum_{i=s_j+1}^{m_j} a_i B(t, T_i)} = \frac{P^{s_j, m_j}_t}{A^{s_j, m_j}_t}, \quad \forall \ t \in [0, T_{s_j}].$$

where we set

$$P^{s_j, m_j}_t = B(t, T_{s_j}) - B(t, T_{m_j}), \quad t \in [0, T_{s_j}],$$

and

$$A^{s_j, m_j}_t = \sum_{i=s_j+1}^{m_j} a_i B(t, T_i), \quad t \in [0, T_{s_j+1}].$$
Let us choose the bond maturing at $T_b$ as a *bond numéraire*. In terms of the *deflated bond prices* $B^b(t,T_i) = B(t,T_i)/B(t, T_b)$, $i = 1, \ldots, n$, we obtain

$$
\kappa_{t}^{s_j, m_j} = \frac{B^b(t, T_{s_j}) - B^b(t, T_{m_j})}{\sum_{i=s_j+1}^{m_j} a_i B^b(t, T_i)} = \frac{P_t^{b,s_j,m_j}}{A_t^{b,s_j,m_j}}, \quad \forall \ t \in [0, T_{s_j} \wedge T_b].
$$

We call the process $A_t^{b,s_j,m_j}$, defined as

$$
A_t^{b,s_j,m_j} = \sum_{i=s_j+1}^{m_j} a_i B^b(t, T_i), \quad t \in [0, T_{s_j+1} \wedge T_b],
$$

the *deflated swap annuity* or *deflated swap numéraire*.
We will deal with the following interrelated modelling issues.

1. Under which assumptions the joint dynamics of a given family of swap rates is supported by an arbitrage-free term structure model, where by a term structure model we mean the joint dynamics of deflated bond prices?

2. How to specify the joint dynamics for a given family of forward swaps in terms of “drifts” and “volatilities” under a single probability measure?

3. Under which assumptions the swap rates and/or CDS spreads follow positive processes and the default time can be constructed?
Each forward swap corresponds to the linear equation in which deflated bond prices are treated as "unknowns".

Specifically, we deal with the following swap equation associated with the forward swap $S_j$ and the numéraire bond $B(t, T_b)$

$$B^b(t, T_{s_j}) - \sum_{i=s_j+1}^{m_j-1} \kappa_{s_j, m_j}^i a_i B^b(t, T_i) - (1 + \kappa_{s_j, m_j}^i) a_{m_j} B^b(t, T_{m_j}) = 0.$$ 

The following inverse problem is of interest: describe all families of forward swaps such that the knowledge of the corresponding family of swap rates is sufficient to uniquely specify the associated family of non-zero (or positive) deflated bond prices for any choice of the numéraire bond.
Let $x_i$ stand for a generic value of the deflated bond price $B^b(t, T_i)$ and let $\kappa_j$ be a generic value of the forward swap rate $\kappa_t^j = \kappa_t^{s_j, m_j}$.

Since $x_i$ and $\kappa_j$ are aimed to represent generic values of the corresponding processes in some stochastic model we have that $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$.

However, if the bond $B(t, T_b)$ is chosen to be the numéraire bond then, by the definition of the deflated bond price, the variable $x_b$ satisfies $x_b = 1$ and thus it is more adequate to consider a generic value $(x_0, \ldots, x_{b-1}, x_{b+1}, \ldots, x_n) \in \mathbb{R}^n$. 
We thus obtain, for every $j = 1, \ldots, l$,

$$\kappa_j = \frac{x_{s_j} - x_{m_j}}{\sum_{i=s_j+1}^{m_j} a_i x_i}.$$ 

For brevity, we write $c_{j,i} = \kappa_j a_i$ and $c_{j,m_j} = (1 + \kappa_j) a_{m_j}$

$$-x_{s_j} + \sum_{i=s_j+1}^{m_j-1} c_{j,i} x_i + c_{j,m_j} x_{m_j} = 0.$$ 

For a given family $S = \{S_1, \ldots, S_l\}$ of forward swaps with tenor $\mathcal{T}$, any fixed $b \in \{0, \ldots, n\}$, and an arbitrary $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$, we thus deal with the linear system $C^b \bar{x}^b = \bar{e}^b$ where $\bar{x}^b = (x_0, \ldots, x_{b-1}, x_{b+1}, \ldots, x_{n})$ is the vector of $n$ unknowns and $\bar{e}^b$ is some vector of zeros and ones.
Cycles

Definition

Two swaps are called *adjacent* if the start date of one of them coincides with the maturity date of another. If distinct swaps $S_{j_1}, \ldots, S_{j_k}$ are such that $S_{j_m}$ is adjacent to $S_{j_m+1}$ for $m = 1, \ldots, k - 1$ then the pair $(T_{s_{j_1}}, T_{m_{j_k}})$ is termed the *path*. A path is called a *cycle* if the equality $T_{s_{j_1}} = T_{m_{j_k}}$ holds.

Lemma

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a family of $n$ swaps with the tenor structure $\mathcal{T}$.  
(i) If there are no cycles in $\mathcal{S}$ then each date from $\mathcal{T}$ is either the start date or the maturity date of some forward swap $S_j$ from $\mathcal{S}$, that is, the equality $\mathcal{T}_0(\mathcal{S}) = \mathcal{T}$ holds.  
(ii) Conversely, if $\mathcal{T}_0(\mathcal{S}) = \mathcal{T}$ then there are no cycles in $\mathcal{S}$. 

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Example

1. Let us consider an example of a family $S$ of forward swaps without a cycle.

2. We assume here that $b = 0$ so that $B(t, T_b) = B(t, T_0)$. Let $\mathcal{T} = \{T_0, \ldots, T_7\}$ and let $S$ be given by the following linear system

$$C^0 \bar{x}^0 = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 & 0 & 0 \\
 c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\
 -1 & c_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & c_{4,2} & c_{4,3} & c_{4,4} & c_{4,5} & c_{4,6} & 0 & 0 \\
 0 & 0 & -1 & c_{5,4} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & c_{6,5} & c_{6,6} & c_{6,7} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & c_{7,7} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7 \end{bmatrix} = \begin{bmatrix} 1 \\
 1 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \end{bmatrix} = \bar{e}^0.$$

3. The swaps $S_1$ and $S_5$ are adjacent and they yield the path $(T_0, T_4)$.
4. The swaps $S_4$ and $S_7$ are adjacent and they yield the path $(T_1, T_7)$.
5. No other swaps are adjacent and thus there are no cycles in $S$. 
Problem (IP.1) Provide necessary and sufficient conditions for a family $S$ of forward swaps under which, for almost every $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$, there exists a unique non-zero solution $(x_0, \ldots, x_{b-1}, x_{b+1}, \ldots, x_n) \in \mathbb{R}^n$ to the linear system $C^b \bar{x}^b = \bar{e}^b$.

Definition

A family $S = \{S_1, \ldots, S_l\}$ of forward swaps associated with the tenor structure $\mathcal{T}$ is $\mathcal{T}$-admissible if for any choice of $b \in \{0, \ldots, n\}$ the following property holds: for almost every $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$ there exists a unique non-zero solution $(x_0, \ldots, x_{b-1}, x_{b+1}, \ldots, x_n) \in \mathbb{R}^n$ to the linear system $C^b \bar{x}^b = \bar{e}^b$ corresponding to $S$. 
Galluccio et al. (2007) introduce the following definition of admissibility of $S$.

**Definition**

We say that a family $S$ of forward swaps associated with $\mathcal{T}$ is *admissible* if the following conditions are satisfied:

(i) the number of forward swaps in $S$ equals $n$, i.e., $l = n$,
(ii) any date $T_i \in \mathcal{T}$ coincides with the reset/settlement date of at least one forward swap from $S$,
(iii) there are no cycles in $S$.

Galluccio et al. (2007) claim that the admissibility of $S$ is equivalent to the existence of a unique non-zero solution to the inverse problem.
Counter-example

1. The following counter-example shows that a family $S$ with a cycle can be $\mathcal{T}$-admissible.

2. Let $n = 3$ and let $(s_j, m_j), j = 1, 2, 3$ be given as $(0, 2), (2, 3)$ and $(0, 3)$, respectively.

3. The swaps $S_1$ and $S_2$ yield the path $(T_0, T_3)$ and this path is also given by the swap $S_3$, so that a cycle $(T_0, T_0)$ exists.

4. For $b = 0$, we obtain the following linear system

$$C^0 x^0 = \begin{bmatrix} c_{1,1} & c_{1,2} & 0 \\ 0 & -1 & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \bar{e}^0.$$

5. A unique non-zero solution exists, for almost every $(\kappa^1, \kappa^2, \kappa^1 3) \in \mathbb{R}^3$. 
We say that a date $T_i$, $i = 1, \ldots, n$ is the \textit{relevant date} for the swap $S_j$ when the term $c_{j,i}$ is non-zero.

In addition, the date $T_0$ is the \textit{relevant date} for the swap $S_j$ if it starts at $T_0$.

Let $\mathcal{T}(S_j)$ stand for the set of all relevant dates for the swap $S_j$ and let $\mathcal{T}_0(S_j) = \{T_{s_j}, T_{m_j}\}$.

We denote $\mathcal{T}_0(S)$ – the set of all start/maturity dates for a family $S$, that is,

$$\mathcal{T}_0(S) = \bigcup_{j=1}^I \mathcal{T}_0(S_j) = \bigcup_{j=1}^I \{T_{s_j}, T_{m_j}\}.$$

$\mathcal{T}(S)$ – the set of all relevant dates (i.e., reset/settlement dates) for a family $S$, that is,

$$\mathcal{T}(S) = \bigcup_{j=1}^I \mathcal{T}(S_j).$$
(\mathcal{T}, b\text{-inadmissibility})

**Definition**

For a fixed \( b \in \{0, \ldots, n\} \), we say that a cycle \( S_c \subset S \) is \((\mathcal{T}, b\text{-inadmissible})\) if the number of dates in \( \mathcal{T}(S_c) \setminus \{T_b\} \) is strictly less than the number of swaps in \( S_c \).

**Lemma**

Let \( S \) be a family of forward swaps containing a \((\mathcal{T}, b\text{-inadmissible})\) cycle \( S_c \) for some \( b \in \{0, \ldots, n\} \). Then the family \( S \) is not \( \mathcal{T}\)-admissible.

**Lemma**

Assume that \( l = n \) and \( \mathcal{T}(S) = \mathcal{T} \). If there is no \((\mathcal{T}, b\text{-inadmissible})\) cycle in \( S \) for any \( b \in \{0, \ldots, n\} \) then for any \( b \in \{0, \ldots, n\} \) there exists a permutation of \( S = \{S_1, \ldots, S_n\} \) such that all entries on the diagonal of \( C^b \) are non-zero.
Sufficient Conditions

Lemma

Assume that $l = n$, the equality $\mathcal{T}(S) = \mathcal{T}$ holds and the graph associated with $S$ is connected, that is, for every $s, m \in \{0, \ldots, n\}$ such that $m < s$ there exist a path $(T_s, T_m)$ generated by $S$. Then either:

(i) $\mathcal{T}_0(S) = \mathcal{T}$ and there are no cycles in $S$, or
(ii) $\mathcal{T}_0(S) \neq \mathcal{T}$ and there are no $(\mathcal{T}, b)$-inadmissible cycle in $S$.

Proposition

Assume that $l = n$, the equality $\mathcal{T}(S) = \mathcal{T}$ holds and the graph associated with $S$ is connected. Then a unique solution to the linear system $C^b\bar{x}^b = \bar{e}^b$ exists and it is non-zero, for almost all $(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n$. Hence the family $S$ of forward swaps is $\mathcal{T}$-admissible.
Density Processes Martingale Measures
By an abstract forward swap we mean the start date $T_{s_j}$, the maturity date $T_{m_j}$ as well a pair $P_{s_j}^{s_j,m_j}, A_{s_j}^{s_j,m_j}$ of processes, where $A_{s_j}^{s_j,m_j}$ is a positive process.

The forward swap rate $\kappa^j$ in an extended forward swap starting at $T_{s_j}$ and maturing at $T_{m_j}$ is defined by the formula

$$\kappa^j_t = \kappa_{t}^{s_j,m_j} = \frac{P_{t}^{s_j,m_j}}{A_{t}^{s_j,m_j}}, \quad \forall t \in [0, T_{s_j}].$$

Let $\mathcal{S} = \{S_1, \ldots, S_l\}$ be a family of extended forward swaps associated with the tenor structure $\mathcal{T}$, that is, such that $\mathcal{T}_0(\mathcal{S}) \subset \mathcal{T}$. 

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The families $P_{s_j, m_j}$ and $A_{s_j, m_j}$ of processes corresponding to $S$ are additive, in the sense that the following equations are satisfied for any cycle $S_c \subset S$

$$
\sum_{j \in S_c^1} A_{t}^{s_j, m_j} = \sum_{k \in S_c^2} A_{t}^{s_k, m_k}
$$

and

$$
\sum_{j \in S_c^1} P_{t}^{s_j, m_j} = \sum_{k \in S_c^2} P_{t}^{s_k, m_k}
$$

where $S_c^1$ and $S_c^2$ are any two paths which produce the cycle $S_c$.

Let us define, for a fixed $d \in \{1, \ldots, l\}$,

$$
A_{t}^{d, s_j, m_j} = \frac{A_{t}^{s_j, m_j}}{A_{t}^{s_d, m_d}}, \quad \forall t \in [0, T_{s_j} \wedge T_{s_d}].
$$

For an arbitrary choice of the swap numéraire $A_{s_d, m_d}$ the family $A_{d, s_j, m_j}, j = 1, \ldots, l$, of swap deflated annuities is additive as well.
Lemma

For any cycle $S_c$ in $S$, the following equality holds

$$\sum_{j \in S_c^1} A^d, s^1_j, m^1_j, \kappa_t^1, s^1_j, m^1_j = \sum_{k \in S_c^2} A^d, s^2_k, m^2_k, \kappa_t^2, s^2_k, m^2_k.$$ 

Problem: Does a family of forward swaps $S = \{S_1, \ldots, S_l\}$ uniquely specifies the family of swap deflated annuities

$$\mathcal{A}^d = \{A^d, s^j, m^j, j = 1, \ldots, d - 1, d + 1, \ldots, l\}$$

for almost all values of $(\kappa^1, \ldots, \kappa^l) \in \mathbb{R}^l$. 
Problem (IP.2) Provide necessary and sufficient conditions for a family $S$ of abstract forward swaps under which, for almost every $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$, there exists a unique non-zero solution $(y_1, \ldots, y_{d-1}, y_{d+1}, \ldots, y_l) \in \mathbb{R}^l$ to the following set of equations: for any cycle $S_c$ in $S$

\[
\sum_{j \in S_c^1} y_j = \sum_{k \in S_c^2} y_k
\]

and

\[
\sum_{j \in S_c^1} \kappa_j y_j = \sum_{k \in S_c^2} \kappa_k y_k.
\]
Example: Swap Rates

1. Take \( n = 2 \) and \( S = \{ S_1, S_2, S_3 \} \) with \((s_j, m_j)\) equal to \((0, 1), (1, 2)\) and \((0, 2)\).

2. For \( b = 0 \) we obtain the following linear system parametrized by \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3\)

\[
C^0 \bar{x}^0 = \begin{bmatrix}
  c_{1,1} & 0 \\
  -1 & c_{2,2} \\
  c_{3,1} & c_{3,2}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  1 \\
  0 \\
  1
\end{bmatrix} = \bar{e}^0.
\]

3. It is easily seen that no solution exists, for almost all \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3\), and thus Problem (IP.1) has no solution.

4. The corresponding Problem (IP.2) has the form, for \( d = 1 \),

\[
1 + y_2 = y_3, \quad \kappa_1 + \kappa_2 y_2 = \kappa_3 y_3.
\]

5. For almost all \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3\), the unique solution reads

\[
y_2 = \frac{\kappa_1 - \kappa_3}{\kappa_3 - \kappa_2}, \quad y_3 = \frac{\kappa_1 - \kappa_2}{\kappa_3 - \kappa_2}.
\]
In the case of forward CDS spreads, Problem (IP.1) should be modified as follows: for almost all \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3\), find a non-zero solution to the linear system

\[
C^0 \tilde{x}^0 = \begin{bmatrix} c_{1,1} & 0 & 0 \\ 0 & -1 & c_{2,3} \\ c_{3,1} & 0 & c_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ \tilde{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \tilde{e}^0.
\]

The unique non-zero solution exists, for almost all \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3\).

The dynamics of the CDS spreads \((\kappa_1, \kappa_2, \kappa_3)\) can be supported by some model in which we will deal with the following set of martingale measures associated swap numéraires

\[
\mathbb{Q} \xrightarrow{d\mathbb{P}^1_{\mathbb{Q}}} \mathbb{P}^1 \xrightarrow{d\mathbb{P}^1_{\mathbb{P}^1}} \mathbb{P}^2 \xrightarrow{d\mathbb{P}^2_{\mathbb{P}^2}} \mathbb{P}^2
\]
Market Models for CDS Spreads
Set-up and Notation

1. Let \((\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})\) be a filtered probability space, where \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}\) is a filtration such that \(\mathcal{F}_0\) is trivial.

2. We assume that the random time \(\tau\) defined on this space is such that the \(\mathbb{F}\)-survival process \(G_t = \mathbb{Q}(\tau > t | \mathcal{F}_t)\) is positive.

3. The probability measure \(\mathbb{Q}\) is interpreted as the risk-neutral measure.

4. Let \(0 < T_0 < T_1 < \cdots < T_n\) be a fixed tenor structure and let us write \(a_i = T_i - T_{i-1}\).

5. We denote \(\tilde{a}_i = a_i/(1 - \delta_i)\) where \(\delta_i\) is the recovery rate if default occurs between \(T_{i-1}\) and \(T_i\).

6. We denote by \(D(t, T)\) the default-free discount factor over the time period \([t, T]\).
Assume first that the interest rate is deterministic.

The *pre-default forward CDS spread* $\kappa_i^t$ corresponding to the single-period forward CDS starting at time $T_{i-1}$ and maturing at $T_i$ equals

$$1 + \tilde{a}_i \kappa_i^t = \frac{\mathbb{E}_Q\left(D(t, T_i)1_{\{T_i > T_{i-1}\}} \mid \mathcal{F}_t\right)}{\mathbb{E}_Q\left(D(t, T_i)1_{\{T_i > T_i\}} \mid \mathcal{F}_t\right)}, \quad \forall \, t \in [0, T_{i-1}].$$

Since the interest rate is deterministic, we obtain, for $i = 1, \ldots, n$,

$$1 + \tilde{a}_i \kappa_i^t = \frac{\mathbb{Q}(\tau > T_{i-1} \mid \mathcal{F}_t)}{\mathbb{Q}(\tau > T_i \mid \mathcal{F}_t)}, \quad \forall \, t \in [0, T_{i-1}],$$

and thus

$$\frac{\mathbb{Q}(\tau > T_i \mid \mathcal{F}_t)}{\mathbb{Q}(\tau > T_0 \mid \mathcal{F}_t)} = \prod_{j=1}^{i} \frac{1}{1 + \tilde{a}_j \kappa_j^t}, \quad \forall \, t \in [0, T_0].$$
We define the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by setting, for every $t \in [0, T]$,

$$\eta_t = \left. \frac{d\mathbb{P}}{d\mathbb{Q}} \right|_{\mathcal{F}_t} = \frac{\mathbb{Q}(\tau > T_n | \mathcal{F}_t)}{\mathbb{Q}(\tau > T_n | \mathcal{F}_0)}.$$

**Lemma**

For every $i = 1, \ldots, n$, the process $Z^{\kappa, i}$ given by

$$Z^{\kappa, i}_t = \prod_{j=i+1}^{n} \left( 1 + \tilde{a}_j z_j \right), \quad \forall t \in [0, T_i].$$

is a $(\mathbb{P}, \mathcal{F})$-martingale.
CDS Martingale Measures

For any $i = 1, \ldots, n$ we define the probability measure $\mathbb{P}^i$ equivalent to $\mathbb{P}$ (and thus also to $\mathbb{Q}$) on $(\Omega, \mathcal{F}_T)$ by setting

$$
\frac{d\mathbb{P}^i}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = Z^i_t = c_i Z^{\kappa,i}_t = \frac{\mathbb{Q}(\tau > T_i)}{\mathbb{Q}(\tau > T_n)} \prod_{j=i+1}^n (1 + \tilde{a}_j \kappa^i_t)
$$

Note that $Z^{\kappa,n}_t = 1$ and thus $\mathbb{P}^n = \mathbb{P}$.

Assume that the PRP holds under $\mathbb{P} = \mathbb{P}^n$ with the $\mathbb{R}^k$-valued spanning $(\mathbb{P}, \mathcal{F})$-martingale $M$.

Then PRP is also valid with respect to $\mathcal{F}$ under any probability measure $\mathbb{P}^i$ for $i = 1, \ldots, n$.

Hence the positive process $\kappa^i_t$ satisfies, for $i = 1, \ldots, n$,

$$
\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_s \sigma^i_s \cdot d\Psi^i(M)_s
$$

where $\sigma^i$ is an $\mathbb{R}^k$-valued, $\mathcal{F}$-predictable process and $\Psi^i(M)$ is the $\mathbb{P}^i$-Girsanov transform of $M$. 
Proposition

Assume that the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}$ with the spanning $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \ldots, M^k)$. Assume that the positive processes $\kappa^i, i = 1, \ldots, n$ are such that the process

$$Z_{t}^{\kappa, i} = \prod_{j=i+1}^{n} (1 + \tilde{a}_j \kappa^j_t)$$

is a $(\mathbb{P}, \mathbb{F})$-martingale for $i = 1, \ldots, n$. Then there exist $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes $\sigma_{i-1}^i$ such that the joint dynamics of processes $\kappa^i, i = 1, \ldots, n$ under $\mathbb{P}$ are given by

$$d\kappa^i_t = \sum_{l=1}^{k} \kappa^i_t \sigma^i_t \, dM^l_t - \sum_{j=i+1}^{n} \frac{\tilde{a}_j \kappa^i_t \kappa^j_t}{1 + \tilde{a}_j \kappa^j_t} \sum_{l,m=1}^{k} \sigma^i_t \sigma^j_t \, d[M^l,c, M^m,c]_t$$

$$- \frac{1}{Z_t^i} \Delta Z_t^i \sum_{l=1}^{k} \kappa^i_t \sigma^i_t \, \Delta M^l_t.$$
Proposition

Let $M = (M^1, \ldots, M^k)$ be an arbitrary $(\mathbb{P}, \mathbb{F})$-martingale and let $\sigma^i, \ i = 1, \ldots, n$ be $\mathbb{R}^k$-valued, locally bounded, $\mathbb{F}$-predictable processes. Assume that the processes $Z^i, \ i = 1, \ldots, n$ are $(\mathbb{P}, \mathbb{F})$-martingales, where

$$Z_t^i = \frac{\prod_{j=i+1}^{n} (1 + \tilde{a}_j \kappa_t^j)}{\mathbb{E}_{\mathbb{P}} \left( \prod_{j=i}^{n-1} (1 + \tilde{a}_j \kappa_t^j) \right)}.$$

Then the joint dynamics of processes $\kappa^i, \ i = 1, \ldots, n$ under $\mathbb{P}$ are given by the previous proposition. For every $i = 1, \ldots, n$, the process $\kappa^i$ is a $(\mathbb{P}^i, \mathbb{F})$-martingale, where the probability measure $\mathbb{P}^i$ is given by

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = c_i \prod_{j=i+1}^{n} (1 + \tilde{a}_j \kappa_t^j).$$
We will now construct a default time $\tau$ consistent with the dynamics of forward CDS spreads. Let us set

$$M^i_{T_{i-1}} = \prod_{j=1}^{i-1} \frac{1}{1 + \tilde{a}_j \kappa_j^{T_{i-1}}} \quad M^i_{T_i} = \prod_{j=1}^{i} \frac{1}{1 + \tilde{a}_j \kappa_j^{T_i}}.$$

Since the process $\tilde{a}_i \kappa^i$ is positive, we obtain, for every $i = 0, \ldots, n$,

$$G_{T_i} := M^i_{T_i} = \frac{M^i_{T_{i-1}}}{1 + \tilde{a}_i \kappa_i^{T_i}} \leq M^i_{T_{i-1}} =: G^i_{T_{i-1}}.$$

The process $G_{T_i} = M^i_{T_i}$ is thus decreasing for $i = 0, \ldots, n$.

We make use of the canonical construction of default time $\tau$ taking values in $\{T_0, \ldots, T_n\}$.

We obtain, for every $i = 0, \ldots, n$,

$$\mathbb{P}(\tau > T_i | \mathcal{F}_{T_i}) = G_{T_i} = \prod_{j=1}^{i} \frac{1}{1 + \tilde{a}_j \kappa_j^{T_i}}.$$
Assume that we are given a model for Libors \((L^1, \ldots, L^n)\) where \(L^i = L(t, T_{i-1})\) and CDS spreads \((\kappa^1, \ldots, \kappa^n)\) in which:

1. The default intensity \(\gamma\) generates the filtration \(F^\gamma\).
2. The interest rate process \(r\) generates the filtration \(F^r\).
3. The probability measure \(Q\) is the spot martingale measure.
4. The \(\mathcal{H}\)-hypothesis holds, that is, \(F \overset{Q}{\hookrightarrow} G\), where \(F = F^r \vee F^\gamma\).
5. The PRP holds with the \((Q, F)\)-spanning martingale \(M\).

Then it is possible to determine the joint dynamics of Libors and CDS spreads \((L^1, \ldots, L^n, \kappa^1, \ldots, \kappa^n)\) under any martingale measure \(Q^i\).
To construct a model we assume that:

1. A martingale $M = (M^1, \ldots, M^\kappa)$ has the PRP with respect to $(\mathbb{P}, \mathcal{F})$.

2. The family of process $Z^i$ given by

$$Z_t^{L,\kappa,i} := \prod_{j=i+1}^{n} (1 + a_j L_t^j)(1 + \tilde{a}_j \kappa_t^j)$$

are martingales on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that there exists a family of probability measure $\mathbb{P}^i$, $i = 1, \ldots, n$ on $(\Omega, \mathcal{F}_T)$ with the density function given by $\frac{d\mathbb{P}^i}{d\mathbb{P}} = c_i Z^{L,\kappa,i}$. 
The dynamics of $L_i^t$ and $\kappa_i^t$ under $\mathbb{P}^n$ with respect to the spanning $(\mathbb{P}, \mathbb{F})$-martingale $M$ are given by

$$dL_i^t = \sum_{l=1}^{k} \xi_i^l \cdot dM_i^l - \sum_{j=i+1}^{n} \frac{a_j}{1 + a_j L_i^t} \sum_{l,m=1}^{k} \xi_i^l \cdot \xi_i^m \cdot d[M_i^l, M_i^m]_t$$

$$- \sum_{j=i+1}^{n} \frac{\tilde{a}_j}{1 + \tilde{a}_j \kappa_i^t} \sum_{l,m=1}^{k} \xi_i^l \cdot \sigma_t^l \cdot \sigma_t^m \cdot d[M_i^l, M_i^m]_t - \frac{1}{Z_t} \Delta Z_t \sum_{l=1}^{k} \xi_i^l \cdot \Delta M_i^l$$

and

$$d\kappa_i^t = \sum_{l=1}^{k} \sigma_t^l \cdot dM_i^l - \sum_{j=i+1}^{n} \frac{a_j}{1 + a_j L_i^t} \sum_{l,m=1}^{k} \sigma_t^l \cdot \xi_i^m \cdot d[M_i^l, M_i^m]_t$$

$$- \sum_{j=i+1}^{n} \frac{\tilde{a}_j}{1 + \tilde{a}_j \kappa_i^t} \sum_{l,m=1}^{k} \sigma_t^l \cdot \sigma_t^m \cdot d[M_i^l, M_i^m]_t - \frac{1}{Z_t} \Delta Z_t \sum_{l=1}^{k} \sigma_t^l \cdot \Delta M_i^l.$$
Let \((\Omega, \mathcal{G}, \mathcal{F}, \mathbb{Q})\) be a filtered probability space, where \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\) is a filtration such that \(\mathcal{F}_0\) is trivial.

We assume that the random time \(\tau\) defined on this space is such that the \(\mathcal{F}\)-survival process \(G_t = \mathbb{Q}(\tau > t | \mathcal{F}_t)\) is positive.

The probability measure \(\mathbb{Q}\) is interpreted as the risk-neutral measure.

Let \(0 < T_0 < T_1 < \cdots < T_n\) be a fixed tenor structure and let us write \(a_i = T_i - T_{i-1}\).

We no longer assume that the interest rate is deterministic.

We denote by \(D(t, T)\) the default-free discount factor over the time period \([t, T]\).
One-Period CDS Spreads

The *one-period forward CDS spread* $\kappa^i_t = \kappa^{i-1, i}$ satisfies, for $t \in [0, T_{i-1}]$,

$$1 + \tilde{a}_i \kappa^i_t = \frac{\mathbb{E}_Q \left( D(t, T_i)1_{\{\tau > T_{i-1}\}} \mid \mathcal{F}_t \right)}{\mathbb{E}_Q \left( D(t, T_i)1_{\{\tau > T_i\}} \mid \mathcal{F}_t \right)}.$$

Let $A^{i-1, i}$ be the *one-period CDS annuity*

$$A^{i-1, i}_t = \tilde{a}_i \mathbb{E}_Q \left( D(t, T_i)1_{\{\tau > T_i\}} \mid \mathcal{F}_t \right)$$

and let

$$P^{i-1, i}_t = \mathbb{E}_Q \left( D(t, T_i)1_{\{\tau > T_{i-1}\}} \mid \mathcal{F}_t \right) - \mathbb{E}_Q \left( D(t, T_i)1_{\{\tau > T_i\}} \mid \mathcal{F}_t \right).$$

Then

$$\kappa^i_t = \frac{P^{i-1, i}_t}{A^{i-1, i}_t}, \quad \forall \ t \in [0, T_{i-1}].$$
Let $A_{t}^{i-2,i}$ stand for the two-period CDS annuity

$$A_{t}^{i-2,i} = \tilde{a}_{i-1} \mathbb{E}_{Q} \left( D(t, T_{i-1}) \mathbbm{1}_{\{\tau>T_{i-1}\}} \mid \mathcal{F}_{t} \right) + \tilde{a}_{i} \mathbb{E}_{Q} \left( D(t, T_{i}) \mathbbm{1}_{\{\tau>T_{i}\}} \mid \mathcal{F}_{t} \right).$$

and let

$$P_{t}^{i-2,i} = \sum_{j=i-1}^{i} \left( \mathbb{E}_{Q} \left( D(t, T_{j}) \mathbbm{1}_{\{\tau>T_{j-1}\}} \mid \mathcal{F}_{t} \right) - \mathbb{E}_{Q} \left( D(t, T_{j}) \mathbbm{1}_{\{\tau>T_{j}\}} \mid \mathcal{F}_{t} \right) \right).$$

The two-period CDS spread $\tilde{\kappa}_{i} = \kappa_{i-2,i}$ is given by the following expression

$$\tilde{\kappa}_{t} = \kappa_{i-2,i} = \frac{P_{t}^{i-2,i}}{A_{t}^{i-2,i}} = \frac{P_{t}^{i-2,i-1} + P_{t}^{i-1,i}}{A_{t}^{i-2,i-1} + A_{t}^{i-1,i}}, \quad \forall \ t \in [0, T_{i-1}].$$
Our aim is to derive the semimartingale decomposition of $\kappa^i, i = 1, \ldots, n$ and $\tilde{\kappa}^i, i = 2, \ldots, n$ under a common probability measure.

We start by noting that the process $A^{n-1,n}_t$ is a positive $(\mathbb{Q}, \mathbb{F})$-martingale and thus it defines the probability measure $\mathbb{P}^n$ on $(\Omega, \mathcal{F}_T)$.

The following processes are easily seen to be $(\mathbb{P}^n, \mathbb{F})$-martingales

$$
\frac{A^{i-1,i}_t}{A^{n-1,n}_t} = \prod_{j=i+1}^n \frac{\tilde{a}_j(\tilde{\kappa}^j_t - \kappa^j_t)}{\tilde{a}_{j-1}(\tilde{\kappa}^{j-1}_t - \tilde{\kappa}^j_t)} = \frac{\tilde{a}_n}{\tilde{a}_i} \prod_{j=i+1}^n \frac{\tilde{\kappa}^j_t - \kappa^j_t}{\kappa^{j-1}_t - \tilde{\kappa}^j_t}.
$$

Given this family of positive $(\mathbb{P}^n, \mathbb{F})$-martingales, we define a family of probability measures $\mathbb{P}^i$ for $i = 1, \ldots, n$ such that $\kappa^i$ is a martingale under $\mathbb{P}^i$. 
For every $i = 2, \ldots, n$, the following process is a $(\mathbb{P}^i, \mathbb{F})$-martingale

$$\frac{A_{t}^{i-2,i}}{A_{t}^{i-1,i}} = \frac{\tilde{a}_{i-1} \mathbb{E}_{\mathbb{Q}} \left( D(t, T_{i-1}) \mathbb{1}_{\{\tau > T_{i-1}\}} \mid \mathcal{F}_t \right) + \tilde{a}_i \mathbb{E}_{\mathbb{Q}} \left( D(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \mid \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}} \left( D(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \mid \mathcal{F}_t \right)}$$

$$= \tilde{a}_{i-1} \left( \frac{A_{t}^{i-2,i-1}}{A_{t}^{i-1,i}} + 1 \right)$$

$$= \tilde{a}_i \left( \frac{\tilde{\kappa}_t^i - \kappa_t^i}{\kappa_{t-1}^i - \tilde{\kappa}_t^i} + 1 \right).$$

Therefore, we can define a family of the associated probability measures $\tilde{\mathbb{P}}^i$ on $(\Omega, \mathcal{F}_T)$, for every $i = 2, \ldots, n$.

It is obvious that $\tilde{\kappa}^i$ is a martingale under $\tilde{\mathbb{P}}^i$ for every $i = 2, \ldots, n$. 
We will summarise the above in the following diagram

\[
\begin{align*}
\mathbb{Q} & \xrightarrow{d\mathbb{P}^n/d\mathbb{Q}} \mathbb{P}^n \xrightarrow{d\mathbb{P}^{n-1}/d\mathbb{P}^n} \mathbb{P}^{n-1} \xrightarrow{d\mathbb{P}^{n-2}/d\mathbb{P}^{n-1}} \ldots \xrightarrow{d\mathbb{P}^2/d\mathbb{P}^{n-1}} \mathbb{P}^2 \xrightarrow{d\mathbb{P}^1/d\mathbb{P}^2} \mathbb{P}^1 \\
\mathbb{\tilde{P}}^n & \downarrow \quad \mathbb{\tilde{P}}^{n-1} \downarrow \quad \ldots \downarrow \quad \mathbb{\tilde{P}}^2 \downarrow
\end{align*}
\]

where

\[
\frac{d\mathbb{P}^n}{d\mathbb{Q}} = A_t^{n-1,n}
\]

\[
\frac{d\mathbb{P}^i}{d\mathbb{P}^{i+1}} = \frac{A_t^{i-1,i}}{A_t^{i,i+1}} = \frac{\tilde{a}_{i+1}}{\tilde{a}_i} \left( \frac{\tilde{\kappa}_t^{i+1} - \kappa_t^{i+1}}{\kappa_t^i - \tilde{\kappa}_t^i} \right)
\]

\[
\frac{d\mathbb{\tilde{P}}^i}{d\mathbb{P}^i} = \frac{A_t^{i-2,i}}{A_t^{i-1,i}} = \tilde{a}_i \left( \frac{\tilde{\kappa}_t^i - \tilde{\kappa}_t^{i-1}}{\kappa_t^i - \tilde{\kappa}_t^i} + 1 \right).
\]
We are in a position to calculate the semimartingale decomposition of \((\kappa^1, \ldots, \kappa^n, \tilde{\kappa}^2, \ldots, \tilde{\kappa}^n)\) under \(P^n\).

It suffices to use the following Radon-Nikodým densities

\[
\frac{dP^i}{dP^n} = \frac{A^{i-1,i}_t}{A^{n-1,n}_t} = \tilde{a}_n \prod_{j=i+1}^{n} \frac{\tilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \tilde{\kappa}_t^j}
\]

\[
\frac{d\tilde{P}^i}{dP^n} = \frac{A^{i-2,i}_t}{A^{n-1,n}_t} = \tilde{a}_n \left( \frac{\tilde{\kappa}_t^i - \kappa_t^i}{\kappa_t^{i-1} - \tilde{\kappa}_t^i} + 1 \right) \prod_{j=i+1}^{n} \frac{\tilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \tilde{\kappa}_t^j}
\]

\[
= \tilde{a}_n \left( \prod_{j=i}^{n} \frac{\tilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \tilde{\kappa}_t^j} + \prod_{j=i+1}^{n} \frac{\tilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \tilde{\kappa}_t^j} \right)
\]

\[
= \tilde{a}_{i-1} \frac{dP^{i-1}}{dP^n} + \tilde{a}_i \frac{dP^i}{dP^n}
\]

Explicit formulae for the joint dynamics of one and two-period spreads are available.
1. The processes $\kappa^1, \ldots, \kappa^n$ and $\tilde{\kappa}^1, \ldots, \tilde{\kappa}^n$ are $\mathbb{F}$-adapted.

2. For every $i = 0, \ldots, n$ the process $Z_{\kappa,i}$

$$Z_{t}^{\kappa,i} = \frac{c_n}{c_i} \prod_{j=i+1}^{n} \frac{\tilde{\kappa}^j_t - \kappa^j_t}{\kappa^{j-1}_t - \tilde{\kappa}^j_t}$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale where $c_1, \ldots, c_n$ are normalizing constants.

3. For every $i = 0, \ldots, n$ the process $Z_{\tilde{\kappa},i}$ given by the formula

$$Z_{\tilde{\kappa},i} = Z_{\kappa,i} + Z_{\kappa,i-1} = \frac{\kappa^{i-1}_t - \kappa^i_t}{\kappa^{i-1}_t - \tilde{\kappa}^i_t} Z_{\kappa,i}$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale.

4. The process $M = (M^1, \ldots, M^k)$ is the $(\mathbb{P}, \mathbb{F})$-spanning martingale.

5. Probability measures $\mathbb{P}^i$ and $\tilde{\mathbb{P}}^i$, $i = 1, \ldots, n$ have the density processes $Z_{\kappa,i}$ and $Z_{\tilde{\kappa},i}$, $i = 1, \ldots, n$. In particular, $\mathbb{P}^n = \mathbb{P}$ since $Z_{\kappa,n} = 1$. 
Lemma

For any $i = 1, \ldots, n$, the process $X^i$ admits the integral representation have that

$$
\kappa_t^i = \int_{(0,t]} \sigma^i_s \cdot d\Psi^i_t(M)_s
$$

and

$$
\tilde{\kappa}_t^i = \int_{(0,t]} \zeta^i_s \cdot d\tilde{\Psi}^i_t(M)_s
$$

where $\zeta^i = (\zeta^i,^1, \ldots, \zeta^i,k)$ is an $\mathbb{R}^k$-valued, predictable process and the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M^l)$ is given by

$$
\Psi^i(M^l)_t = M^l_t - \left[ (\ln Z^i)^c, M^l,^c \right]_t - \sum_{0<s\leq t} \frac{1}{Z^i_s} \Delta Z^i_s \Delta M^l_s.
$$

An analogous formula yields $\tilde{\Psi}^i(M^l)$. 
**Proposition**

The semimartingale decomposition of the \((\mathbb{P}^i, \mathbb{F})\)-spanning martingale \(\Psi^i(M)\) under the probability measure \(\mathbb{P}^n = \mathbb{P}\) is given by, for \(i = 1, \ldots, n\),

\[
\Psi^i(M)_t = M_t - \sum_{j=i+1}^{n} \int_{(0,t]} \left( \frac{\kappa^{j-1}_s - \kappa^j_s}{\tilde{\kappa}^j_s - \kappa^j_s} \right) \zeta^j_s \cdot d[M^c]_s - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\sigma^j_s \cdot d[M^c]_s}{\tilde{\kappa}^j_s - \kappa^j_s} - \sum_{0 < s \leq t} \frac{1}{Z^i_s} \Delta Z^i_s \Delta M_s.
\]

An analogous formula holds for \(\tilde{\Psi}^i(M)\). Hence the joint dynamics of the process \((\kappa^1, \ldots, \kappa^n, \tilde{\kappa}^2, \ldots, \tilde{\kappa}^n)\) under \(\mathbb{P} = \mathbb{P}^n\) are explicitly known.
Towards Generic Swap Models
Towards Generic Swap Models

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space. We are given a family of swap rates \(S = \{\kappa^1, \ldots, \kappa^l\}\) and a family of \((\mathbb{P}, \mathcal{F})\)-martingales \(\{Z^1, \ldots, Z^l\}\) such that:

1. For each \(j = 1, \ldots, l\), the process \(\kappa^j\) is a positive special semimartingale.
2. For each \(j = 1, \ldots, l\), the process \(\kappa^j Z^j\) is a \((\mathbb{P}, \mathcal{F})\)-martingale.
3. For each \(j = 1, \ldots, l\), \(Z^j\) is uniquely expressed as a \(C^2\) function of some subset of swaps in \(S\). That is the process \(Z^j = f_j(\kappa^{n_1}, \ldots, \kappa^{n_k})\) where \(f_j\) is a \(C^2\) function in variables belonging to \(S_j := \{\kappa^{n_1}, \ldots, \kappa^{n_k}\} \subset S\).
1. Assumption 1 forces the semi-martingale decomposition of $\kappa^j$ to be uniquely determined.

2. Assumption 2 gives the existence of a family of probability measures $\{\mathbb{P}^1, \ldots, \mathbb{P}^l\}$, for which $\kappa^j$ is a martingale under $\mathbb{P}^j$.

3. Assumption 3 together with the fact that the process $Z^j$ is a $(\mathbb{P}, \mathcal{F})$-martingale implies that $Z^j$ has the following integral representation

$$Z^j_t = \sum_{i=n_1}^{n_k} \int_{[0,t]} \frac{\partial f^j_i}{\partial x_i}(\kappa_{s1}^{n_1}, \ldots, \kappa_{sk}^{n_k}) d(\kappa^i)^m_s,$$

where $(\kappa^i)^m$ stands for the (unique) martingale part of $\kappa^i$. 
We claim that the semi-martingale decomposition of a swap rate process $\kappa^n \in S$ can be chosen under $\mathbb{P}$ by choosing a family of “volatility” processes.

Hence $\kappa^j = N^j \in \mathcal{M}(\mathbb{P}^j, \mathbb{F})$ and by inverse Girsanov’s transform the martingale part of $\kappa^n$ must have the following representation under $\mathbb{P}$

$$(\kappa^j)^m = N^j_t - \int_{(0,t]} Z^j_s d\left[\frac{1}{Z^j}, N^j\right]_s = N_t$$

or, equivalently, the semi-martingale decomposition of $\kappa^n$ under $\mathbb{P}$ is given by

$$\kappa^j = N^j_t = N_t + \int_{(0,t]} Z^j_s d\left[\frac{1}{Z^j}, N^j\right]_s$$

where $N$ is unique since the Girsanov transform is a bijection.
For the purpose of modelling, $N^j_t$ is defined under $\mathbb{P}^j$ as follows

$$N^j_t = \int_0^t \kappa^j_s \sigma^j_s \cdot d\Psi^j(M)_s.$$ 

Therefore, specifying $N^j_t$ is equivalent to specifying the “volatility” $\sigma^j$.

The martingale part of $\kappa^j$ can be expressed as

$$(\kappa^j)^m = \int_0^t \kappa^n_s \sigma^n_s \cdot d\Psi^j(M)_s - \int_{(0,t]} Z^j_s \kappa^n_s \sigma^n_s \cdot d\left[ \frac{1}{Z^j_s}, d\Psi^j(M)_s \right]_s = \int_0^t \kappa^n_s \sigma^n_s \cdot dM^j_s$$

where $M^j$ is a $(\mathbb{P}, \mathbb{F})$-martingale.

The process $Z^j$ has the following decomposition

$$Z^j_t = \sum_{i=n_1}^{n_k} \int_{[0,t)} \frac{\partial f_j}{\partial x_i}(\kappa^n_{s_1}, \ldots, \kappa^n_{s_k}) \kappa^n_{s_i} \sigma^n_{s_i} \cdot dM^n_{s_i}.$$ 

Choice of processes $Z^j$ is equivalent to the choice of the family of “volatilities”.

We conclude that the choice of “volatilities” specifies the model.