NUMERICAL COMPUTATION OF AN INVERSE CONTACT PROBLEM IN ELASTICITY

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Abstract. In this paper we develop a numerical computational method for solving an inverse contact problem in elasticity. The inverse contact problem is a classical ill-posed problem in the sense of Hadamard. Applying the Tikhonov regularization technique to transform the problem into an optimization problem, we successfully obtain a numerical approximation to the solution by using standard finite element method. Based on the conditional stability estimate for the inverse problem, we also give a convergence analysis on the numerical approximation.

1. Introduction

Let \( R^3_+ \equiv \{(x_1, x_2, x_3) \in R^3 | x_3 > 0\} \) be the half space and \( \Omega \) be a bounded domain on the boundary \( \{x_3 = 0\} \).

We consider the following boundary value problem for Lamè equation:

\[
\begin{cases}
Lu \equiv \Delta u + \lambda \nabla(\nabla \cdot u) = 0, \\
(B_1 u)(x_1, x_2, 0) = -G(x_1, x_2), \quad (x_1, x_2) \in R^2,
\end{cases}
\]

where \( u = (u_1, u_2, u_3) \) denotes the displacement vector and \( |\lambda| \neq 1 \) is a non-trivial constant whose value depends on the Lamè constant. The boundary condition is

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prescribed by the operator $B_1$:

\begin{equation}
B_1 = \begin{pmatrix}
\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\
0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\
(\lambda - 1) \frac{\partial}{\partial x_3} & (\lambda - 1) \frac{\partial}{\partial x_2} & (\lambda + 1) \frac{\partial}{\partial x_3}
\end{pmatrix}
\end{equation}

and the surface stress $g = g(x_1, x_2)$ on $x_3 = 0$ by:

\begin{equation}
G(x_1, x_2) = \begin{pmatrix} 0 \\ 0 \\ g(x_1, x_2) \end{pmatrix}.
\end{equation}

If the surface stress $g$ is known, then the displacements vector $u$ can simply be computed as a direct problem. The inverse contact problem to be investigated in this paper is to determine the contact domain $\Omega$ defined as

\begin{equation}
\Omega = \{(x_1, x_2) \mid g(x_1, x_2) \neq 0\}
\end{equation}

from the unknown values $g(x_1, x_2)$. For illustration purpose, assume that the contact domain $\Omega$ lies on the boundary $\{x_3 = 0\}$ and the surface stress on the contact domain is not known. The main problem is then to determine the domain $\Omega$ and $g(x_1, x_2), (x_1, x_2) \in \Omega$ from the values of $u_3(x_1, x_2, 0), (x_1, x_2) \in D \subset \Omega^c$, where $D$ is an open set.

This problem is a classical inverse contact problem. It is known from the basic theories on partial differential equations and elasticity that the forward contact problem is well-posed in the Hadamard’s sense. The inverse contact problem, however, is severely ill-posed in the Hadamard’s sense. In fact, in general we cannot guarantee the continuous dependence of the solution on the given data. That is, if we do not assume any a-priori bound on the solution, a small change of the initial data may induce dramatic variation of the solution. Please refer to Kress [9] and Tikhonov et al. [15] for details on the treatment of ill-posed problems.

We follow the approaches in Cheng et al. [2, 3] to transform the inverse contact problem into a Fredholm integral equation of the first kind. Based on the conditional stability estimate given in their papers, we give a convergence analysis on the approximated solution.
This paper will be organized as: In Section 2, we give a formal mathematical formulation of the inverse contact problem. Following the idea given in Cheng et al. [2, 3], the inverse problem is first transformed to a first kind integral equation with analytical kernel. Since the integral equation is still a severely ill-posed problem, we apply the Tikhonov regularization technique [15] to obtain a conditional stability estimate on the approximation to the solution in Section 3. In Section 4, the convergence analysis on the proposed numerical approximation is given. A numerical algorithm is then devised in Section 5 with numerical verifications given in Section 6.

2. Formulation of the problem

Let $D$ be a bounded domain in $\mathbb{R}^2$ satisfying $D \cap B_\rho = \emptyset$ where $B_\rho = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x| < \rho\}$. Let $W_{\text{loc}}^n(\Omega)$ and $W_0^n(\Omega)$ be the usual Sobolev spaces. Denote $L^p(\Omega) = W_{\text{loc}}^0(\Omega)$, $H^n(\Omega) = W_{\text{loc}}^n(\Omega)$ and

$$
\mathcal{H} = \{v(x_1, x_2, x_3) \in H_{\text{loc}}^2(\mathbb{R}^3_+) \mid v(\cdot, \cdot, x_3) \in L^2(\mathbb{R}^2) \text{ for almost all } x_3 > 0, \text{ess sup}_{x_3 > 0} \|v(\cdot, \cdot, x_3)\|_{L^2(\mathbb{R}^2)} < \infty\}. 
$$

From Fainburd et al. [7] and Kolakowski [8], the displacement vector $u = (u_1, u_2, u_3) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H}$ satisfies the Lamé equation in $\mathbb{R}^3_+ = \{x_3 > 0\}$:

$$
(2.1) \quad Lu = \Delta u + \lambda \nabla(\nabla \cdot u) = 0,
$$

and the boundary condition:

$$
(2.2) \quad (B_1 u)(x_1, x_2, 0) = -G(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2,
$$

which prescribes the surface stress $g = g(x_1, x_2)$ on $x_3 = 0$. From Cheng et al. [2]), there exists a unique solution $u = u(g)$ to the equations (2.1) – (2.2).

Without loss of generality, throughout this paper, $g$ is assumed to be compactly supported.

From the theory of elasticity (refer, for example, Qian [12]), it is known that the contact domain $\Omega$ on the boundary $\{x_3 = 0\}$ can be defined as
(2.3) \[ \Omega = \Omega(g) = \{ x = (x_1, x_2) | g(x_1, x_2) \neq 0 \} . \]

Assume a domain \( D \subset \Omega^c = \mathbb{R}^2 \setminus \Omega \). The inverse contact problem is then the determination of the contact domain \( \Omega \) and the stress \( g \) from the measurement of the surface displacement \( u_3 \) outside the contact domain. That is, the problem is to determine the contact domain \( \Omega \) and \( g(x) \) on \( \Omega \) from

\[ (2.4) \quad f(x_1, x_2) = u_3(x_1, x_2, 0), \quad (x_1, x_2) \in \Omega \subset \Omega^c. \]

Remark 2.1. It follows from \( D \subset \Omega^c \) that \( g(x) = 0, \quad x \in D \). That is, the solution of the inverse contact problem leads to an over-determining boundary data of \( u \) on \( D: B_1 u(= 0) \) and \( u_3 \).

Throughout this paper, we fix \( \rho > 0 \) arbitrarily and denote

\[ (2.5) \quad U = \{ \varphi \in L^2(\mathbb{R}^2) | \Omega(\varphi) \subset B_{\rho_1} \subset B_{\rho} \}, \]

where \( \rho_1 \) and \( \rho \) are fixed positive constants and \( \rho_1 < \rho \). It is assumed that \( g \in U \).

Remark 2.2. It is clear that if the unknown stress \( g \) is determined, then the contact domain \( \Omega \) can be determined from (2.3). The inverse contact problem is a kind of non-destructive evaluation problems which is also severely ill-posed in the Hadamard’s sense.

The following section will give a conditional stability on the inverse contact problem.

3. Conditional Stability

First we summarize from Cheng et al. [2, 3] some required results on the uniqueness and ill-posedness of the inverse contact problem:

**Theorem 3.1.** (Uniqueness) The contact domain \( \Omega \) and the stress vector \( g \) can be uniquely determined from the given data \( u_3|_D \).
Theorem 3.2. (Ill-posedness) The inverse contact problem is severely ill-posed in the Hadamard sense to determine the $g \in L^2(\Omega)$ from $f \in L^2(D)$. That is, there exists a $g_0 \in U$ and a sequence $g_n \in U$, $n = 1, 2, \cdots$ such that $u_3(g_n) \rightarrow u_3(g_0)$ in $L^2(D)$ but $g_n$ does not converge to $g_0$.

In Cheng et al. [2, 3], some global and local conditional stability estimations were obtained. Since the Banach norms are used in these papers, it is not so easy to perform numerical computation. In the following, we will give the conditional stability estimations under the Hilbert norms. Although we require slightly stronger assumptions in the stability proof, the proof is easier to be understood and verified. The assumptions can also be weakened if required.

We first need the following lemmas:

Lemma 3.3. The inverse contact problem (2.1) - (2.4) is equivalent to determine the unknown function $g(y)$ that satisfies the following integral equation:

$$
\int_{\Omega} \frac{g(y)}{|x-y|} \, dy = \frac{4\pi \lambda}{\lambda + 1} f(x), \quad x \in D.
$$

Here, $g(x) = 0$, $x \in \Omega^c$.

Define a harmonic function $G(x, z)$ in $\mathbb{R}^4 \setminus \hat{\Omega}$, where $\hat{\Omega} = \{(x, z) \mid x \in \Omega, z = 0\}$, by

$$
G(x, z) = \int_{\Omega} \frac{g(y)}{\sqrt{|x-y|^2 + z^2}} \, dy.
$$

It is easy to verify that

1. $\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2}\right) G(x, z) = 0$ for $(x, z) \in \mathbb{R}^4 \setminus \hat{\Omega}$,
2. $\lim_{z \to 0^+} \frac{\partial G(x, z)}{\partial z} = -2\pi g(x)$ for $x \in \Omega$,
3. $\lim_{z \to 0^+} \frac{\partial G(x, z)}{\partial z} = 0$ for $x \in \Omega^c$.

Without of loss generality, we assume that $D \subset B_{2\rho}$.

Notice that the ball $\tilde{B}_{2\rho} = \{(x, z) \mid |x|^2 + z^2 < 4\rho^2\}$. Using the standard techniques in analysis, we can obtain the following lemma:
Lemma 3.4. \( G(x, z) \) is harmonic on \( \tilde{B}_{2\rho} \setminus \hat{\Omega} \) and satisfies

\[
\| G(x, z) \|_{H^2(\partial \tilde{B}_{2\rho})} \leq C_1, \\
\| G(x, z) \|_{H^2(D)} \leq C_1,
\]

where \( C_1 > 0 \) is a constant which depends only on the values of \( \| g \|_{H^2(\Omega)} \) and \( \rho \).

Finally, consider the following Cauchy problem for the Laplace equation

\[
\begin{cases}
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) G(x, z) = 0, & (x, z) \in \tilde{B}_{\rho} \setminus \hat{\Omega}, \\
G(x, 0) = f(x), & x \in D, \\
\frac{\partial G}{\partial z}(x, 0) = 0, & x \in D.
\end{cases}
\]

From the conditional stability results for the Cauchy problem for the Laplace equations (refer [5, 6, 11]), we now have the following theorem:

**Theorem 3.5.** \((L^2\text{-conditional stability})\) Suppose that \( g \in U \) satisfies \( g \in H^2_0(\Omega) \) and \( \| g \|_{H^2(\Omega)} \leq M \). There exists a positive constant \( C \) which depends on \( M, \rho_1 \) and \( \rho \), but is independent of \( g \), such that

\[
\| g \|_{L^2(\Omega)} \leq C \frac{1}{\log \left( \frac{\rho}{\rho_1} \right)},
\]

where \( \epsilon = \| f \|_{L^2(D)} \) and \( 0 < \epsilon < 1 \).

The following section then give a convergence analysis based on the conditional stability result given in Theorem 3.5.

4. Convergence analysis

Denote \( v = \frac{1+1}{4\pi x} g \). The integral equation (3.1) can be formulated in the following operator form:

\[
\mathcal{A}v = f,
\]

where \( \mathcal{A}v(x) = \int_{\Omega} \phi(x, y)v(y)dy, x \in D \) and \( \phi(x, y) = \frac{1}{|x-y|^2} \). Suppose that \( \overline{\Omega} \cap \overline{D} = \emptyset \). The operator \( \mathcal{A} \) is a continuous map from \( L^2(\Omega) \) to \( L^2(D) \). The kernel \( \phi \) is then analytic on \( \Omega \times D \) and hence the inverse problem is severely ill-posed (refer Kress [9]).
Assume that the given measurement data \( f_\delta \) of \( f \) contains an error \( \delta \) in the sense \( \| f - f_\delta \|_{L^2(D)} \leq \delta \). We apply the Tikhonov regularization technique to the equation (4.1). For any fixed \( \delta > 0 \) and \( v \in H^2_0(\Omega) \), we define the following Tikhonov functional:

\[
F_\alpha(v) := \| Av - f_\delta \|_{L^2(D)}^2 + \alpha \| v \|_{H^2(\Omega)}^2,
\]

where \( \alpha \) is a positive regularization parameter.

Since \( F_\alpha(v) \geq 0 \), there exists a \( \beta > 0 \) such that

\[
\beta = \inf_{v \in H^2_0(\Omega)} F_\alpha(v).
\]

Let \( v^\delta_\alpha \in H^2_0(\Omega) \) such that

\[
F_\alpha(v^\delta_\alpha) \leq \beta + \delta^2.
\]

This function \( v^\delta_\alpha \) is a regularized solution for (4.1). It is obvious that such a \( v^\delta_\alpha \) exists.

The following theorem gives the convergence rate of the regularized solution.

**Theorem 4.1.** Suppose that \( v_0 \in H^2_0(\Omega) \) is the exact solution of equation (4.1), i.e., \( \mathcal{A}v_0 = f \). Choose \( \alpha = \alpha(\delta) = \delta^2 \), then the regularized solution \( v^\delta_\alpha \) converges to \( v_0 \) at the following convergence rate:

\[
\| v^\delta_\alpha - v_0 \|_{L^2(\Omega)} \leq C |\log \delta|,
\]

where \( C > 0 \) is a constant which depends only on \( v_0 \).

**Proof:** The proof follows from Theorem 3.5 and Theorem 2.1 in Cheng and Yamamoto [4].

5. Numerical algorithm

For numerical illustration, we consider in the following a special case of the inverse contact problem with rectangular contact and measurement domains:

\[
\mathcal{A}v(s, t) = \int_a^b \int_c^d \frac{v(x, y)}{\sqrt{(s - x)^2 + (t - y)^2}} \, dx \, dy = f(s, t), \quad (s, t) \in (a', b') \times (c', d'),
\]
where $\Omega = (a, b) \times (c, d), D = (a', b') \times (c', d')$ and $\overline{\Omega} \cap \overline{D} = \emptyset$.

From the definition of regularized solution and the given data $f_\delta$, we can obtain an approximated solution to equation (5.1) by minimizing the functional (4.2) on a finite element space. Unlike standard finite element method which requires generating meshes on the original coordinate system $(x, y)$, our proposed computation requires also a generation of mesh on the coordinate system $(s, t)$.

It is easy to generate a rectangular subdivision over the rectangular domains $\Omega$ and $D$ by drawing lines parallel to the x-axis and y-axis. Denote $n$ and $n'$ to be the total numbers of nodes and $m$ and $m'$ to be the total numbers of elements on the domains $\overline{\Omega}$ and $\overline{D}$ respectively. Here, $(x_i, y_i)$ denotes the coordinate of node $i$ on $\Omega$, $i = 1, 2, \cdots, n$, and $(s_i, t_i)$ is the coordinate of node $i$ on $D$, $i = 1, 2, \cdots, n'$.

In the following computations, the bi-cubic Hermite elements are chosen (details please refer Brenner et al.[14]). The shape functions defined on the local and global coordinate system are denoted respectively by $\hat{N}^c_{i}(\xi, \eta)$ and $N^c_{i}(x, y)$ for $i = 1, 2, \cdots, 12$.

The Hermite interpolation of $v$ on each element $e$ is then

$$v^c_{i}(x, y) = \sum_{i=1}^{12} v^c_{i} \cdot N^c_{i}(x, y), \quad (x, y) \in e,$$

where $v^c_{i}, v^c_{4+i}, v^c_{8+i}$ and $v^c_{12+i}$ are respectively the values of $v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial xy}$ on node $i$ over element $e$, $i = 1, 2, 3, 4$. The Hermite interpolation function $v_h$ on the domain $\Omega$ is then given as

$$v_h = v^c_{i}, \quad (x, y) \in e \quad \text{for all } e \subset \Omega.$$

The interpolation function $v_h$ of (5.3) can also be expressed as the following global form:

$$v_h = \sum_{i=1}^{4n} v_i \varphi_i,$$

where $\varphi_i, \varphi_{n+i}, \varphi_{2n+i}$ and $\varphi_{3n+i}$ are the basis functions defined on node $i$ which are piecewise polynomials derived by the standard scheme in finite element method and $v_i, v_{n+i}, v_{2n+i}$, $v_{3n+i}$ are the values of the unknown functions $v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial xy}$ on the global node $i$. 
The finite element subspace is defined to be
\[ V_h = \text{span}\{\varphi_i, \quad i = 1, \cdots, 4n\}. \]

It is well known from Reddy [13] that the discretized finite element subspace satisfies:
\[ V_h \subset H^2(\Omega). \]

Denote
\[ V^0_h = \{ v \mid v \in V_h, \quad v|_{\partial \Omega} = \partial v/\partial n|_{\partial \Omega} = 0 \}. \]

We then have \( V^0_h \subset H^2_0(\Omega) \). The discretized problem is then to find \( v^\delta_{\alpha,h} \in V^0_h \) such that
\[
F_\alpha(v^\delta_{\alpha,h}) = \min_{v_h \in V^0_h} F_\alpha(v_h).
\]

As usual, we ignore the boundary condition arising in the space \( V^0_h \) and discrete the functional (4.2) over the finite element space \( V_h \). A boundary constraint will be imposed later for the solution of the resultant linear system.

We use classical scheme to assemble the stiffness matrix contributed by \( \|v_h\|_{H^2(\Omega)} \).

Denote this stiff matrix \( K_v = (k_{ij}^v) \), \( i, j = 1, 2, \cdots, 4n \).

In the following we derive the stiffness matrix contributed by the functional \( \|A v - f\delta\|_{L^2(D)} \). Notice that
\[
\|A v_h - f\delta\|_{L^2(D)}^2 = \sum_{i,j=1}^{4n} [v_i v_j (A \varphi_i, A \varphi_j)_{L^2(D)}] + \sum_{i=1}^{4n} v_i (A \varphi_i, f\delta)_{L^2(D)} + (f\delta, f\delta)_{L^2(D)},
\]
where \( (A \varphi_i, A \varphi_j)_{L^2(D)} \) is the entry of the stiffness matrix induced by the functional \( \|A v - f\delta\|_{L^2(D)} \) and \( (A \varphi_i, f\delta)_{L^2(D)} \) is the component of force vector.

For the computation of \( (A \varphi_i, A \varphi_j)_{L^2(D)} \), we need to calculate \( A \varphi_i(s, t) \) by:
\[
A \varphi_i(s, t) = \sum_{k=1}^{l_i} \int_{e_{ik}} \int_{e_{ik}} \phi(s, t; x, y) N_{iik}^{e_{ik}} (x, y) dx dy, 
\]
\[
= \sum_{k=1}^{l_i} \int_{e_{ik}} \int_{e_{ik}} \phi(s, t; x, y) N_{iik}^{e_{ik}} (x, y) \left[\frac{\partial (e_{ik})}{4} d\xi d\eta, \right.
\]
\[
= \sum_{k=1}^{l_i} \sum_{p, q=1}^{4} \phi(s, t; x, y) N_{iik}^{e_{ik}} (x, y) \left[\frac{\partial (e_{ik})}{4} w_p w_q \right].
\]
where \( l_i \) is the total number of elements connected with node \( i \); \( e_{ik} \) is the \( k \)-th element connected with node \( i \); \( N_{ii_k}^{ei_k} \) is the \( ii_k \)-th shape function on element \( e_{ik} \); where \( ii_k \) is the local number of node \( i \) on element \( e_{ik} \) and \( \hat{N}_{ii_k}^{ei_k} \) is the corresponding shape function in the reference element. Here, \((x_e^{ei_k}, y_e^{ei_k})\) is the coordinate of the center of each element \( e_{ik} \); \( x_{h_e^{ei_k}} \) and \( y_{h_e^{ei_k}} \) are the step sizes of the element \( e_{ik} \) along the x-axis and y-axis respectively; \( s(e_{ik}) \) is the area of element \( e_{ik} \); \((\xi_p, \eta_q)\), \( p, q = 1, 2, 3, 4 \), are the coordinates of the 16 points used in the Gauss integration and \( w_p, w_q \) are the weights of the Gauss integration.

In order to calculate each integration \((A\varphi_i, A\varphi_j)_{L^2(D)}\), we sub-divide the domain \( D \) into rectangular elements and then transform them to a standard element on the natural coordinate system. Finally, the Gauss integration formula enable us to obtain the approximated values for the components of the stiffness matrix and the corresponding force vector.

For illustration, consider the following calculation

\[
k_{ij} = (A\varphi_i, A\varphi_j)_{L^2(D)},
\]

\[
= \sum_{k=1}^{m'} \int \int_{e_k} A\varphi_i A\varphi_j(s, t) ds dt,
\]

\[
= \sum_{k=1}^{m'} \int_{-1}^{1} \int_{-1}^{1} \frac{s(e_k)}{4} A\varphi_i A\varphi_j \left( s_{e_k}^c + \frac{1}{2} sh_{e_k}^c \xi, t_{e_k}^c + \frac{1}{2} th_{e_k}^c \eta \right) d\xi d\eta,
\]

\[
= \sum_{k=1}^{m'} \sum_{p=q=1}^{4} \frac{s(e_k)}{4} A\varphi_i A\varphi_j \left( s_{e_k}^c + \frac{1}{2} sh_{e_k}^c \xi_p, t_{e_k}^c + \frac{1}{2} th_{e_k}^c \eta_q \right) w_p w_q,
\]

where \( e_k \) is any element on domain \( D \) and \((s_{e_k}^c, t_{e_k}^c)\) is the coordinate of the center of the element \( e_k \), \( sh_{e_k}^c \) and \( th_{e_k}^c \) are the step sizes of the element \( e_k \) along the s-axis and t-axis respectively. The other components can be obtained similarly as

\[
b_i = (A\varphi_i, f_\delta)_{L^2(D)}.
\]

Hence, the stiffness matrix induced by the functional \( \| A v - f_\delta \|_{H^1}^2 \) becomes \( K_A = (k_{ij}) \) and the force vector of the finite element equations is \( B = (b_i) \).

Thus the Tikhonov functional (4.2) becomes

\[
F_\alpha(v_h) = V^T K_\alpha V + B^T V + (f_\delta, f_\delta)_{L^2}, \quad K_\alpha = K_A + \alpha K_V,
\]

(5.6)
where $V = [v_1, v_2, \cdots, v_{4n}]^T$ and $v_h = \sum_{k=1}^{4n} v_k \varphi_k \in V_h$. Consider the boundary condition of the function in space $V_0^h$, we use the general method given in Li [10] to impose a boundary constraint to the discrete functional (5.6). Suppose that $k$ is an index of a node on $\partial \Omega$ and $v_h|_k = \partial_n v_h|_k = 0$, we can get a new functional by changing the stiffness matrix and force vector as follows:

\begin{equation}
F_\alpha(v_h) = V^T K_\alpha V + B^T V + (f_\delta, f_\delta)_{L^2},
\end{equation}

where $K_\alpha = (\bar{e}_{ij})$ and $B = (\bar{b}_i)$ given by

\begin{equation}
\bar{e}_{ij} = \begin{cases} 1, & i = j = k, k + n, \\ 0, & i \neq j, i, j = k, k + n, \\ k_{ij}, & \text{elsewhere}, \end{cases}
\end{equation}

\begin{equation}
\bar{b}_i = \begin{cases} 0, & i = k, k + n, \\ b_i, & i \neq k, k + n. \end{cases}
\end{equation}

This transformation is then applied to all boundary nodes. The final stiffness matrix and force vector are still denoted to be $K_\alpha$ and $B$. The final form of the functional is the same as given in (5.7).

It is well known that the minimizer $v_{\delta,h}^\alpha$ can be obtained by solving the following finite element equations:

\[ K_\alpha V^\alpha = B, \]

where $V^\alpha = [v_1^\alpha, v_2^\alpha, \cdots, v_{4n}^\alpha]^T$. Therefore, an approximated solution for the integral equation (5.1) is given as

\[ v_{\delta,h}^\alpha = \sum_{k=1}^{4n} v_k^\alpha \varphi_k. \]

When taking $\alpha = \delta^2$, the corresponding regularized solution is

\[ v_{\delta,h}^\alpha = \sum_{k=1}^{4n} v_k^{\alpha(\delta)} \varphi_k. \]

Let $T^h$ be the rectangular subdivision of $\Omega$ and $h$ is the length of the longest side of the rectangular $T \in T^h$. It is known from Susanne et al. [14] that for any
given function \( g \in H^2 \), its interpolation \( I_h g = \sum_{i=1}^{4n} g_i \phi_i \) converges to \( g \) as \( h \to 0 \). Moreover, if \( g \in H^m \) for \( m \geq 2 \), we have

\[
\|g - I_h g\|_{H^2} \leq Ch^{m-2} \|g\|_{H^m}.
\]

Since \( F_\alpha(v) \) is strongly convex and locally Lipschitz continuous (Bruckner and Cheng [1]), there is a unique \( v^\delta_{\alpha(\delta)} \in H^2_0 \) such that

\[
F_\alpha(\delta)(v^\delta_{\alpha(\delta)}) = \inf_{v \in H^2_0} F_\alpha(\delta)(v),
\]

and a unique \( v^\delta_{\alpha(\delta),h} \in V^0_h \) satisfying

\[
F_\alpha(\delta)(v^\delta_{\alpha(\delta),h}) = \inf_{v_h \in V^0_h} F_\alpha(\delta)(v_h).
\]

From equation (5.10), the continuity of \( F_\alpha \) implies

\[
F_\alpha(\delta)(v^\delta_{\alpha(\delta),h}) \to F_\alpha(\delta)(v^\delta_{\alpha(\delta)}) \quad \text{as} \quad h \to 0 \quad \text{for fixed} \quad \delta.
\]

Therefore, for sufficiently small size, say \( h < h(\delta) \), \( v^\delta_{\alpha(\delta),h} \) is a regularized solution in the sense of (4.3).

6. Numerical examples

For numerical verification, two numerical examples are given in this section to verify the accuracy and efficiency of the proposed method.

Suppose that the contact domain \( \Omega \) is \([0,1] \times [0,1]\) and the measurement domain \( D \) is \([2,3] \times [2,3]\). By taking an exact solution \( v \), we calculate an approximated \( f_\delta \) to right hand side \( f \) of the integral equation (4.1) by using Gauss integration formula with 16 points. The error level \( \delta \) is decided by the subdivision of the domain \( \Omega \), the smoothness of the exact solution and the kernel function \( \phi(s, t; x, y) \) on the variables \( x, y \). Using \( f_\delta \) and the proposed numerical method, we compute the regularized solution \( v_h \) and compare it with the exact \( v \). In our computations, the total number of elements on \( \Omega \) is taken to be \( 20 \times 20 \). Here, we also sub-divide the measurement domain \( D \) into \( 2 \times 2 \) rectangular elements for performing the two-dimensional integration on \( D \).

In the following, the regularized solution \( v^\delta_{\alpha(\delta),h} \) is simply denoted by \( v^*_h \). Consider the following examples.
Example 1: As an exact solution of (4.1), take \( v = x^2(1-x)^2y^2(1-y)^2 \). The numerical comparison between the regularized solution \( v_h^* \), \( h = 0.05 \) and the exact solution \( v \) is shown in Figure 1. Curve plots of \( v_h^*(x, jh) \) and \( v(x, jh) \) for \( h = 0.05 \), \( j = 1, 4, 7, 10 \) are displayed in Figure 2. The high accuracy indicates that our proposed method is accurate and effective.

Furthermore, we include a random error of order \( 10^{-3} \) into the values of \( f_\delta \). The numerical results between \( v_h^* \) and \( v \) on line \( y = jh, j = 1, 4, 7, 10 \) are shown in Figure 3. It is observed even for noisy data, the numerical results are still accurate.

Example 2: Let \( v \) be \((x - 0.1)^2(0.9 - x)^2(y - 0.1)^2(0.9 - y)^2\) for \((x, y) \in \Omega_1 = [0.1, 0.9] \times [0.1, 0.9]\) and 0 for \((x, y) \in \Omega \setminus \Omega_1\). The numerical results for the regularized solution \( v_h^* \), \( h = 0.05 \) and the exact solution \( v \) are shown in Figure
Figure 2. Curve plots of \( v^*_h(x, jh) \) and \( v(x, jh) \) for \( h = 0.05 \) and \( \delta \approx 0, \alpha = 10^{-15} \). Solid line represents the exact solution and dotted line represents its approximation.

4. Curve plots of \( v^*_h(x, jh) \) and \( v(x, jh) \) for \( h = 0.05, j = 1, 4, 7, 10 \) are displayed in Figure 5. We observe that the support of the solution \( v \) can be approximately reconstructed in the case of exact input data. For the case with noisy data (noisy level \( \delta = 10^{-3} \)), the numerical results in Figure 6 are, however, quite inaccurate. This indicates that the numerical computation is very sensitive to the perturbation of input data.

7. Conclusion

In this paper we apply the standard finite element method to solve an inverse contact problem in elasticity. This inverse contact problem is a classical ill-posed problem in the sense of Hadamard. Applying the Tikhonov regularization technique
to transform the problem into an optimization problem, then we successfully obtain a stable numerical approximation to the solution by choosing a priori regularization parameter. Based on the conditional stability estimate for the inverse problem, we also obtain a convergence order estimate for the numerical approximation.

References

Figure 4. (a) Surface plot of $v^*_h(x, y)$ for $h = 0.05$ and $\delta \approx 0$, $\alpha = 10^{-20}$. (b) Exact solution $v(x, y)$ for Example 2.

Figure 5. Curve plots of $v_h^*(x,jh)$ and $v(x,jh)$ for $h = 0.05$ and $\delta \approx 0, \alpha = 10^{-20}$. Solid line represents the exact solution and dotted line represents its approximation.

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Figure 6. Curve plots of $v^*(x,jh)$ and $v(x,jh)$ for $h = 0.05$ and $\delta = 10^{-3}$, $\alpha = 10^{-6}$. Solid line represents the exact solution and dotted line represents its approximation.