Confidential
The Work of Curt McMullen

Steve Smale

Department of Mathematics
City University of Hong Kong
Kowloon, Hong Kong

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Curt McMullen has been awarded the Fields Medal for his work in dynamics as well as for his contributions to the theory of computation, complex variables, geometry of three manifolds, and other areas of mathematics.

I limit myself here to a discussion of part of his contributions to the theory of algorithms and two of his main results in dynamics.

The search for understanding of solutions of a polynomial equation has had a central and glorious place in the history of mathematics. Already the ancient Greek mathematicians had approximated the square root of two, ie., the solution of $x^2 = 2$ by what is now called Newton's Method. Providing a solution for equations such as $x^2 + 1 = 0$ led to the introduction of complex numbers in mathematics. Group theory was introduced to understand which polynomial equations could be solved in terms of radicals. Earlier there had been such formulas for degrees 2 (the quadratic formula taught in high school), 3 and 4. For degrees greater than 4 there are no such formulae.

Instead of formulae, algorithms have been developed which produce (perhaps by complex routines) a sequence of better and better approximations to a solution of a general polynomial equation. In the most satisfactory case, iteration of a single map, Newton's Method, converges for almost all polynomials and initial points to a solution of a given quadratic equation; it is a "generally convergent algorithm". But for degree 3 polynomials it converges too infrequently.

Thus I was led to raise the question as to whether there existed such a generally convergent algorithm which succeeds for polynomial equations of each degree.

Curt McMullen answers this question in his thesis, under Dennis Sullivan, where he shows that no such algorithm exists for polynomials of degree greater than 3, and for polynomials of degree 3 he produces a new algorithm which does converge to a solution for almost all polynomials and initial points.

Thus McMullen "finished the job" since this work answers, in degree 3, "yes", and degree greater than three, "no"; it is complete. This indicates his depth of understanding of the situation and is characteristic of his later work.

T. Y. Li and Jim Yorke introduced the word "chaos" into dynamics in connection with the map of population biology, $L_r : [0,1] \to [0,1], L_r(x) = rx(1-x)$. Bob May had been intrigued by this map because there was an infinite sequence of period doubling parameters r_i converging to $s = 3.57....$

Soon thereafter Mitch Feigenbaum's work demonstrating the universality properties of this map helped establish the acceptance by physicists of the new discipline of dynamical systems. The sequence $(r_i - r_{i-1)}/(r_{i+1} - r_i))$ has a limit, a number which is independent of the period doubling map! Key to Feigenbaum's work were the concepts of renormalization and an infinite sequence of renormalizations of an iterate of L_s converging to a fixed point of the renormalization operator on an appropriate function space of maps. Renormalization occurs when the restriction of an iterate is similar to the original map.

Lanford found computer assisted proofs of these conjectures of Feigenbaum and then Sullivan put them into a broader, detailed, conceptual framework, finding important relations between 1-dimensional dynamics and parts of classical function theory as Kleinian groups.

Yet the proof of fast (exponential) convergence of the renormalizations, a basic ingredient in this program, was still missing until McMullen's beautifull work was published in the second of his two Annals of Math Studies in 1996. The fast convergence was necessary to yield the crucial rigidity of the theory ("$C^{1+\alpha}$ conjugacy").

I will state another result of McMullen, proved in the first of these two Studies.

Complex one dimensional dynamics is the study of the iterates of a polynomial map $P : \mathbb{C} \to \mathbb{C}$. This has become the most advanced and the most technical part of dynamics. Yet one simple problem may be singled out as giving some focus to this subject.

Among polynomial maps of a given degree d, are the hyperbolic ones dense?

A polynomial is called hyberbolic (sometimes axiom A), if the orbits of its critical points tend under time to an attracting cycle.

I naively gave this as a thesis problem in the 1960's. Today it is still unsolved even for $d = 2$, but there are a number of partial results.

Quadratic dynamics may be studied for polynomials in the normalized form $P_c(z) = z^2 + c$, with parameter $c \in \mathbb{C}$. The unique critical point is zero and if it tends to ∞ under iteration, the dynamics is well understood in terms of symbolic dynamics. The Mandelbrot set M is defined as the set of $c \in \mathbb{C}$ for which this is not the case. This often pictured set can be thought of as a "tree with fruit", the fruit being the components of its interior. McMullen's theorem asserts:

If c is in a component of the interior of the Mandelbrot set which meets the real axis, then P_c is hyperbolic.

As McMullen writes, "if one runs the real axis through M, then all the fruit which is skewered is good".

Earlier Yoccoz had done an important special case, and I am ignoring here much other earlier fundamental work in complex (and real) dynamics such as Fatou, Julia, Douady and Hubbard. I am also ignoring the later work of Lyubich and Graczyk-Swiatek.

Moreover in these two books McMullen establishes new results in complex function theory and the geometry of 3-manifolds.

I have given a brief glimpse of what Curt McMullen has accomplished, but would like to emphasize that his work has encompassed a large realm of the kind of mathematics that lies at the cross-section of many paths of our rich culture. McMullen is not a dynamicist, not an analyst nor a geometer. He is a mathematician.