Some Lower Bounds for the Complexity of Continuation Methods.

Jean-Pierre Dedieu *
LAO, Université Paul Sabatier
31062 Toulouse Cedex 04, France.
email : dedieu@cict.fr

Steve Smale
Department of Mathematics,
City University of Hong Kong,
83 Tat Chee Avenue, Hong Kong,
email : masmale@math.cityu.edu.hk

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1. Introduction and Main Results.

In this note we consider the zero finding problem for a homogeneous polynomial system

$$f : \mathbb{C}^{n+1} \to \mathbb{C}^m$$

with $m \leq n$, $f = (f_1, \ldots, f_m)$, $f_i \in \mathcal{H}_{d_i}$, the space of homogeneous polynomials $f_i : \mathbb{C}^{n+1} \to \mathbb{C}$ with $degree(f_i) = d_i$. The well-determined ($m = n$) and underdetermined ($m < n$) cases are considered together. We also let $D = \max d_i$, $d = (d_1, \ldots, d_m)$ and $\mathcal{H}_d = \mathcal{H}_{d_1} \times \ldots \times \mathcal{H}_{d_m}$.

The projective Newton method has been introduced by M. Shub in [6] and is defined by

$$N_f(x) = x - \left(Df(x)|_{x^\perp}\right)^{-1} f(x)$$

when $m = n$, with $Df(x)|_{x^\perp}$ the restriction at $x^\perp$ of the derivative of f at x. Here $x^\perp$ denotes the space orthogonal to x in $\mathbb{C}^{n+1}$. When $m \leq n$, we take

$$N_f(x) = x - \left(Df(x)|_{x^\perp}\right)^{\dagger} f(x)$$

where, for any linear operator $A : \mathbb{E} \to \mathbb{F}$ between two Hermitian spaces, $A^\dagger$ denotes its Moore-Penrose inverse. $A^\dagger = A^\star (AA^\star)^{-1}$ when A is onto, which is the case considered here.

A Newton continuation method sequence (NCM sequence) is a sequence of pairs

$$(f_i, \zeta_i) \in (\mathcal{H}_d)^\star \times (\mathbb{C}^{n+1})^\star, \quad 0 \leq i \leq k,$$

(given a vector space E, $E^\star$ denotes the set of nonzero vectors) satisfying the following conditions :

$$f_i(\zeta_i) = 0, \quad 0 \leq i \leq k,$$

and

$$\alpha(f_{i+1}, \zeta_i) \leq \alpha_0, \quad \textit{with associated zero } \zeta_{i+1}, \quad 0 \leq i \leq k-1.$$

This last condition, we will make it precise later, implies that the projective Newton's sequence

$$x_0 = \zeta_i, \quad x_{p+1} = N_{f_{i+1}}(x_p), \quad p \geq 0,$$

converges quadratically towards ζ_{i+1}.

The complexity of an NCM sequence (f_i, ζ_i), $0 \leq i \leq k$, is measured by k. Upper bounds for the complexity of NCM sequences have been given by M. Shub and S. Smale in their papers [7], [8], [9], about the complexity of Bézout's Theorem. They give an upper bound depending mainly on the degree D of the considered system and on the condition number of the homotopy. The case of sparse polynomial systems is studied in [2] by J.-P. Dedieu, the case of homogeneous polynomial systems by G. Malajovich [5] and L. Blum, F. Cucker, M. Shub, S. Smale in their book [1], Chap. 14, the case of multihomogeneous underdetermined polynomial systems by J.-P. Dedieu and M. Shub [3] and the case of overdetermined polynomial systems by J.-P. Dedieu and M. Shub [4].

Our main results here are two lower bounds for the complexity of a NCM sequence. In the first one we relate this complexity to the degree :

Theorem 1. *For any NCM sequence* (f_i, ζ_i), $0 \le i \le k$, *one has*

$$k \ge c \max(1, \frac{D-1}{2}) d_R(\zeta_0, \zeta_k),$$

where $c > 0$ *is a universal constant given below and* $d_R(\zeta_0, \zeta_k)$ *the Riemannian distance in* $\mathbb{P}(\mathbb{C}^{n+1})$ *between* ζ_0 *and* ζ_k.

The Riemannian distance in $\mathbb{P}(\mathbb{C}^{n+1})$ is defined by

$$d_R(u, v) = \arccos \frac{|< u, v >|}{\|u\| \|v\|}$$

for any u, $v \in (\mathbb{C}^{n+1})^\star$.

Remark 1. This bound is sharp and this complexity is obtained for the family of systems defined by

$$f_{t,j}(z_0, z_1, \ldots, z_n) = z_0^{d_j - 1}(z_j - \zeta_{t,j} z_0), \ 1 \le j \le n,$$

where $\zeta_t = (1-t)(1, 0, \ldots, 0) + t(1, a_1, \ldots, a_n)$, $a \in \mathbb{C}^n$ given.

For any $f \in \mathcal{H}_d$ let us define

$$\Sigma_f = \{x \in (\mathbb{C}^{n+1})^\star \quad : \quad rank\ Df(x) < m\}.$$

In our second theorem, we give a lower bound for the complexity of an NCM sequence in terms of the arithmetic mean of the distances of ζ_i from Σ_{f_i}.

Theorem 2. *For any NCM sequence* (f_i, ζ_i), $0 \le i \le k$, *one has*

$$k \ge c \frac{d_R(\zeta_0, \zeta_k)}{k^{-1} \sum_{i=1}^{k} d_R(\zeta_i, \Sigma_{f_i})},$$

where $c > 0$ *is (another) universal constant.*

Remark 2. This lower bound shows that the complexity of an NCM sequence increases with the proximity of singular points. This proximity is measured here by the arithmetic mean of the distances $d_R(\zeta_i, \Sigma_{f_i})$, $1 \le i \le k$.

Corollary 1. *Let* $\epsilon > 0$ *be given. For any NCM sequence* (f_i, ζ_i), $0 \le i \le k$, *such that*

$$d_R(\zeta_i, \Sigma_{f_i}) \le \epsilon, \quad 1 \le i \le k$$

we have

$$k \geq c\epsilon^{-1} d_R(\zeta_0, \zeta_k),$$

with c as in Theorem 2.

The proofs of these theorems are based on Smale's alpha-theory introduced by S. Smale in [10]. We use here its homogeneous version as described in J.-P. Dedieu-M. Shub [3]. When $Df(x)$ is onto, we define

$$\gamma(f,x) = \max\left(1, \|x\| \max_{k\geq 2} \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}\right),$$

$$\beta(f,x) = \|x\|^{-1} \|(Df(x)|_{x^\perp})^\dagger f(x)\|,$$

$$\alpha(f,x) = \beta(f,x)\gamma(f,x).$$

In the definition of $\gamma(f,x)$, $\| \quad \|$ is the operator norm with respect to the canonical Hermitian structure over $\mathbb{C}^{n+1}$.

These three quantities are invariant under scaling and under unitary transformations:

$$\star(f,x) = \star(f,\lambda x) = \star(\lambda f, x) = \star(f o u, u^{-1}(x)),$$

with $\star \in \{\alpha, \beta, \gamma\}$, for any $x \in (\mathbb{C}^{n+1})^\star$, $\lambda \in \mathbb{C}^\star$ and any unitary transformation u in $\mathbb{C}^{n+1}$. When $Df(x)$ is not onto, we take

$$\alpha(f,x) = \beta(f,x) = \gamma(f,x) = \infty.$$

The following theorem ([3], Theorem 1) justifies our definition of a NPC sequence:

Theorem 3. *There is a universal constant $\alpha_0 > 0$ with the following property : for any homogeneous system $f \in \mathcal{H}_d$ and $x \in (\mathbb{C}^{n+1})^\star$, if $\alpha(f,x) \leq \alpha_0$, then the projective Newton sequence*

$$x_0 = x, \quad x_{k+1} = N_f(x_k)$$

is defined and satisfies

$$\|x_{k+1} - x_k\| / \|x_k\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \beta(f,x)$$

for any $k \geq 0$. This sequence converges to a zero $\zeta \in (\mathbb{C}^{n+1})^\star$ of f and

$$d_R(\zeta, x_k) \leq \sigma \left(\frac{1}{2}\right)^{2^k - 1} \beta(f,x)$$

with

$$\sigma = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{2^i - 1} = 1.6328\ldots$$

We can take $\alpha_0 = 1/137$.

2. Proofs of Theorems 1 and 2.

These proofs are the consequences of the three following propositions. In the first one we compute the minimum value of $\gamma(f,\zeta)$.

Proposition 1. *We have*

$$\max(1, \frac{D-1}{2}) = \min \gamma(f,\zeta),$$

where the minimum is taken over all pairs $(\zeta, f) \in (\mathbb{C}^{n+1})^\star \times (\mathcal{H}_d)^\star$ *with* $f(\zeta) = 0$.

Proof. We first prove that $(D-1)/2$ is a lower bound for $\gamma(f,\zeta)$. We can suppose that *rank* $Df(\zeta) = m$ since, otherwise, $\gamma(f,\zeta) = \infty$. Using the invariance properties of $\gamma(f,\zeta)$ under scaling and unitary transformations, we also can suppose that $\zeta = (1, 0, \ldots, 0)$. Since $f(\zeta) = 0$ we have

$$f_i(z) = z_0^{d_i - 1}(a_{i,1}z_1 + \ldots + a_{i,n}z_n) + g_i(z), \quad 1 \le i \le m,$$

with *degree* $(g_i, z_0) \le d_i - 2$. Let us denote by A the $m \times n$ matrix with entries $a_{i,j}$. Thus $Df(\zeta) = (0|A)$. The second derivative of $f = (f_1, \ldots, f_m)$ is given by

$$D^2 f_i(\zeta) = (d_i - 1)\begin{pmatrix} 0 & A_i \\ A_i^T & O_n \end{pmatrix} + D^2 g_i(\zeta),$$

where $A_i = (a_{i,1}, \ldots, a_{i,n})$ and O_n is the $n \times n$ zero matrix. Since *degree* $(g_i, z_0) \le d_i - 2$, for any $v \in \mathbb{C}^{n+1}$ we have

$$D^2 f_i(\zeta)(\zeta, v) = (d_i - 1)\sum_{j=1}^{n} a_{i,j} v_j$$

so that

$$D^2 f(\zeta)(\zeta, v) = Diag(d_i - 1)A\tilde{v}, \quad \tilde{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Here $Diag(d_i - 1)$ denotes the $m \times m$ diagonal matrix with diagonal enties $d_i - 1$. This gives

$$\left(Df(\zeta)|_{\zeta^\perp}\right)^\dagger D^2 f(\zeta)(\zeta, v) = \begin{pmatrix} 0 \\ A^\dagger Diag(d_i - 1)A\tilde{v} \end{pmatrix}$$

and consequently

$$\|A^\dagger Diag(d_i-1)A\|/2 = \|\left(Df(\zeta)|_{\zeta^\perp}\right)^\dagger D^2f(\zeta)/2\| \le \gamma(f,\zeta).$$

Let us consider the matrix $B = A^\dagger Diag(d_i-1)A$. Let $A = U\Sigma V$ be a singular value decomposition of A : U and V are unitary $m\times m$ and $n\times n$ matrices and $\Sigma = (\Delta|0)$ with $\Delta = Diag(\sigma_i)$, $\sigma_1 \ge \ldots \ge \sigma_m > 0$ the singular values of A. We have

$$B = V^\star \begin{pmatrix} \Delta^{-1} \\ 0 \end{pmatrix} U^\star Diag(d_i-1)U(\Delta|0)V = V^\star \begin{pmatrix} \Delta^{-1}U^\star Diag(d_i-1)U\Delta & 0 \\ 0 & 0 \end{pmatrix} V$$

so that the eigenvalues of B are $d_1-1,\ldots,d_m-1$ and 0. Since $\|B\| \ge \rho(B)$ (its spectral radius) we get

$$\|B\| = \|A^\dagger Diag(d_i-1)A\| \ge D-1$$

and this proves the inequality

$$\gamma(f,\zeta) \ge \max(1, \frac{D-1}{2}).$$

In order to prove the converse inequality, we study the following example :

$$f_i(z) = z_0^{d_i-1}z_i, \quad 1 \le i \le m, \quad z = (z_0,\ldots,z_n),$$

and we consider $\zeta = (1,0,\ldots,0)$, so that $f(\zeta) = 0$. The derivative of f is given by

$$Df(\zeta) = (0, I_m, O_{n-m}),$$

with I_m the $m\times m$ identity matrix and O_{n-m} the $(n-m)\times(n-m)$ zero matrix. The other derivatives are given by

$$D^kf_i(\zeta)(u^1,\ldots,u^k) = \sum_{\substack{0\le i_j\le n \\ 1\le j\le k}} \frac{\partial^k f_i(\zeta)}{\partial z_{i_1}\ldots\partial z_{i_k}} u^1_{i_1}\ldots u^k_{i_k},$$

for any $u^1,\ldots,u^k \in \mathbb{C}^{n+1}$. These partial derivatives are equal to 0 except when, for some $j = 1\ldots k$, we have

$$i_1 = \ldots = i_{j-1} = i_{j+1} = \ldots = i_k = 0 \ \ and \ \ i_j = i.$$

In this case, this partial derivative is equal to $(d_i-1)\ldots(d_i-k+1)$. Thus

$$D^kf_i(\zeta)(u^1,\ldots,u^k) = \sum_{j=1}^k (d_i-1)\ldots(d_i-k+1)u_0^1\ldots u_0^{j-1}u_i^ju_0^{j+1}\ldots u_0^k$$

and

$$\|D^k f(\zeta)(u^1,\dots,u^k)\| = \left(\sum_{i=1}^m \left|\sum_{j=1}^k (d_i-1)\dots(d_i-k+1)u_0^1\dots u_0^{j-1}u_i^j u_0^{j+1}\dots u_0^k\right|^2\right)^{1/2}$$

$$\leq (D-1)\dots(D-k+1)\sum_{j=1}^k\left(\sum_{i=1}^m\left|u_0^1\dots u_0^{j-1}u_i^j u_0^{j+1}\dots u_0^k\right|^2\right)^{1/2}$$

$$\leq (D-1)\dots(D-k+1)k/2$$

when $\|u^1\| = \dots = \|u^k\| = 1$, as it can be proved by induction over $k \geq 2$. Consequently

$$\|\zeta\| \max_{k\geq 2}\left\|(Df(\zeta)|_{\zeta^\perp})^\dagger \frac{D^k f(\zeta)}{k!}\right\|^{\frac{1}{k-1}} \leq \max_{k\geq 2}\left(\frac{1}{2}\binom{D-1}{k-1}\right)^{\frac{1}{k-1}} = \frac{D-1}{2}$$

using the fact that the sequence $\left(\frac{1}{2}\binom{D-1}{k-1}\right)^{\frac{1}{k-1}}$, $k \geq 2$, is decreasing. This yealds $\gamma(f,\zeta) \leq \max(1,(D-1)/2)$ and achieves the proof of Proposition 1. $\square$

Proposition 2. *There is a universal constant $c > 0$ with the following property. For any $f \in (\mathcal{H}_d)^\star$, ζ and $x \in (\mathbb{C}^{n+1})^\star$, if $\alpha(f,x) \leq \alpha_0$ with associated zero ζ then*

$$d_R(\zeta,x)\gamma(f,\zeta) \leq c.$$

Such a proposition appears in [10]. We give here a similar result in the context of homogeneous systems. It is a consequence of the three following lemmas.

Let us introduce the function

$$\psi(u) = 2u^2 - 4u + 1, \quad 0 \leq u \leq 1 - \frac{\sqrt{2}}{2}.$$

This function is decreasing from 1 at $u = 0$ to 0 at $u = 1 - \sqrt{2}/2$. We first start with a linear algebra lemma. Its proof may be found in [3], Lemma 2.a.

Lemma 1. *Let X and Y be Hermitian spaces and $A, B : X \to Y$ linear operators with B onto. If*

$$\|B^\dagger(B-A)\| \leq \lambda < 1$$

then A is onto and

$$\|A^\dagger B\| < \frac{1}{1-\lambda}.$$

Lemma 2. *Let x and $y \in (\mathbb{C}^{n+1})^\star$ be given such that $Df(x)|_{x^\perp}$ is onto, $\|x\| = 1$ and $u = \|y - x\|\gamma(f, x) < 1 - \frac{\sqrt{2}}{2}$. Then $Df(y)|_{x^\perp}$ is onto and*

$$\|(Df(y)|_{x^\perp})^\dagger Df(x)|_{x^\perp}\| \le \frac{(1-u)^2}{\psi(u)}.$$

Proof. $Df(y) = Df(x) + \sum_{k\ge 2} k\frac{D^k f(x)}{k!}(y-x)^{k-1}$ so that

$$(Df(x)|_{x^\perp})^\dagger(Df(y)|_{x^\perp} - Df(x)|_{x^\perp}) = \sum_{k\ge 2} k(Df(x)|_{x^\perp})^\dagger \frac{D^k f(x)}{k!}(y-x)^{k-1}|_{x^\perp}.$$

If we take the operator norm of both sides we get

$$\|(Df(x)|_{x^\perp})^\dagger(Df(y)|_{x^\perp} - Df(x)|_{x^\perp})\| \le \sum_{k\ge 2} k\gamma(f,x)^{k-1}\|y-x\|^{k-1} =$$

$$\sum_{k\ge 2} ku^{k-1} = \frac{1}{(1-u)^2} - 1$$

and this number is < 1 since $u < 1 - \frac{\sqrt{2}}{2}$. By Lemma 1 $Df(y)|_{x^\perp}$ is onto and

$$\|(Df(y)|_{x^\perp})^\dagger Df(x)|_{x^\perp}\| \le \frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1\right)} = \frac{(1-u)^2}{\psi(u)}. \quad \square$$

Lemma 3. *Let x and $\zeta \in (\mathbb{C}^{n+1})^\star$ be given such that $\|x\| = \|\zeta\| = 1$ and $\alpha(f, x) \le \alpha_0$ with associated zero ζ. Then*

$$\gamma(f,\zeta) \le \frac{\gamma(f,x)}{(1-\sigma\alpha_0)\psi(\sigma\alpha_0)}$$

where σ and α_0 are the constants appearing in Theorem 3.

Proof. Since $f(\zeta) = 0$ we have $\mathbb{C}\zeta \subset ker Df(\zeta)$ so that $(Df(\zeta)|_{\zeta^\perp})^\dagger = Df(\zeta)^\dagger$ is the minimum norm right inverse of $Df(\zeta)$. Thus

$$\gamma(f,\zeta) = \max\left(1, \max_{k\ge 2}\left\|(Df(\zeta)|_{\zeta^\perp})^\dagger \frac{D^k f(\zeta)}{k!}\right\|^{\frac{1}{k-1}}\right) \le$$

$$\max\left(1, \max_{k\ge 2}\left\|(Df(\zeta)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!}\right\|^{\frac{1}{k-1}}\right).$$

Since $\alpha(f,x) \le \alpha_0$, $Df(x)|_{x^\perp}$ is onto so that

$$\left\| (Df(\zeta)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\| \le \left\| (Df(\zeta)|_{x^\perp})^\dagger Df(x) \right\| \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|.$$

Let us denote $u = \|x - \zeta\| \gamma(f,x)$. Since $\alpha(f,x) \le \alpha_0$, by Theorem 3, we have $d_R(\zeta, z) \le \sigma\beta(f,x)$. Moreover, since $\|x\| = \|\zeta\| = 1$, we also have $\|x - \zeta\| \le d_R(\zeta, z)$ so that

$$u \le d_R(\zeta, x)\gamma(f,x) \le \sigma\alpha(f,x) \le \sigma\alpha_0 < 1 - \frac{\sqrt{2}}{2}.$$

Thus, by Lemma 2

$$\left\| (Df(\zeta)|_{x^\perp})^\dagger Df(x) \right\| < \frac{(1-u)^2}{\psi(u)}.$$

We also have

$$\left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\| \le \sum_{l \ge 0} \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^{k+l} f(x)}{k! l!} \right\| \|x - \zeta\|^l$$

$$\le \sum_{l \ge 0} \| \frac{(k+l)!}{k! l!} \gamma(f,x)^{k+l-1} \|x - \zeta\|^l = \frac{\gamma(f,x)^{k-1}}{(1-u)^{k+1}}.$$

Thus

$$\gamma(f,\zeta) \le \max \left(1, \max_{k \ge 2} \left(\frac{(1-u)^2}{\psi(u)} \frac{\gamma(f,x)^{k-1}}{(1-u)^{k+1}} \right)^{\frac{1}{k-1}} \right)$$

and, using the inequality $u \le \sigma\alpha_0$ we are done. □

Proof of Proposition 2. Since $\alpha(f,x) \le \alpha_0$, by Theorem 3, we have $d_R(\zeta, x) \le \sigma\beta(f,x)$. Using Lemma 3 we obtain

$$d_R(\zeta, x)\gamma(f,\zeta) \le \sigma\beta(f,x) \frac{\gamma(f,x)}{(1-\sigma\alpha_0)\psi(\sigma\alpha_0)} \le \frac{\sigma\alpha_0}{(1-\sigma\alpha_0)\psi(\sigma\alpha_0)}$$

and we are done. □

Our last ingredient is a corollary of the following theorem (gamma-theorem for homogeneous polynomial systems) see [3], Theorem 2, and [1], Chap. 14, Theorem 1, for the case $m = n$.

Theorem 4. *There is a universal constant γ_o with the following property : let $\zeta \in (\mathbb{C}^{n+1})^\star$ be a zero of $f \in (\mathcal{H}_d)^\star$ and $x \in (\mathbb{C}^{n+1})^\star$. If*

$$\|x - \zeta\| \gamma(f,\zeta) / \|\zeta\| \le \gamma_0$$

then the projective Newton sequence

$$x_0 = x, \quad x_{k+1} = N_f(x_k),$$

is defined and converges to a zero $\zeta' \in (\mathbb{C}^{n+1})^\star$ *of* f *and*

$$d_R(\zeta', x_k) \le \sigma \left(\frac{1}{2}\right)^{2^k-1} \beta(f, x).$$

Proposition 3. *For any* $\zeta \in (\mathbb{C}^{n+1})^\star$ *and any* $f \in (\mathcal{H}_d)^\star$ *such that* $f(\zeta) = 0$ *we have*

$$d_R(\zeta, \Sigma_f)\gamma(f, \zeta) \ge \gamma_0.$$

Proof. If $x \in \Sigma_f$ then the projective Newton sequence $x_0 = x$, $x_{k+1} = N_f(x_k)$, is not defined. By Theorem 4 we have necessarily

$$\|x - \zeta\|\gamma(f, \zeta)/\|\zeta\| > \gamma_0.$$

When $< x, \zeta >= 0$ then $d_R(\zeta, x) = \pi/2$ and the conclusion holds. When $< x, \zeta >\neq 0$, scaling x such that $< x - \zeta, x >= 0$, we obtain

$$d_R(x, \zeta) \ge \|x - \zeta\|/\|\zeta\|$$

and we are done. □

Proofs of Theorems 1 and 2. Given an NCM sequence (f_i, ζ_i), $0 \le i \le k$, since $\alpha(f_{i+1}, \zeta_i) \le \alpha_0$ with associated zero ζ_{i+1}, we get by Proposition 2

$$d_R(\zeta_{i+1}, \zeta_i)\gamma(f_{i+1}, \zeta_{i+1}) \le c.$$

By Proposition 1 we obtain

$$\max(1, \frac{D-1}{2})d_R(\zeta_{i+1}, \zeta_i) \le c,$$

so that

$$\max(1, \frac{D-1}{2})d_R(\zeta_0, \zeta_k) \le \max(1, \frac{D-1}{2})\sum_{i=0}^{k-1} d_R(\zeta_{i+1}, \zeta_i) \le ck$$

and this proves Theorem 1.

As previously we have

$$d_R(\zeta_{i+1}, \zeta_i)\gamma(f_{i+1}, \zeta_{i+1}) \le c$$

and by Proposition 3

$$d_R(\zeta_{i+1},\zeta_i)\gamma_0 d_R(\zeta_{i+1},\Sigma_{f_{i+1}})^{-1} \le d_R(\zeta_{i+1},\zeta_i)\gamma(f_{i+1},\zeta_{i+1}) \le c$$

so that

$$d_R(\zeta_0,\zeta_k) \le \sum_{i=0}^{k-1} d_R(\zeta_{i+1},\zeta_i) \le c\gamma_0^{-1}k\left(\frac{1}{k}\sum_{i=0}^{k-1} d_R(\zeta_{i+1},\Sigma_{f_{i+1}})\right)$$

and this proves Theorem 2. □

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